

High-Frequency Asymptotics for Lipschitz-Killing Curvatures of Excursion Sets on the Sphere

Domenico Marinucci* and Sreekar Vadlamani

Department of Mathematics, University of Rome Tor Vergata and
Tata Institute for Fundamental Research, Bangalore

March 12, 2013

Abstract

In this paper, we shall be concerned with geometric functionals and excursion probabilities for some nonlinear transforms evaluated on Fourier components of spherical random fields. In particular, we consider both random spherical harmonics and their smoothed averages, which can be viewed as random wavelet coefficients in the continuous case. For such fields, we consider smoothed polynomial transforms; we focus on the geometry of their excursion sets, and we study their asymptotic behaviour, in the high-frequency sense. We put particular emphasis on the analysis of Euler-Poincaré characteristics, which can be exploited to derive extremely accurate estimates for excursion probabilities. The present analysis is motivated by the investigation of asymmetries and anisotropies in Cosmological data.

- Keywords and Phrases: High-Frequency Asymptotics, Spherical Random Fields, Gaussian Subordination, Lipschitz-Killing Curvatures, Minkowski Functionals, Excursion Sets.
- AMS Classification: 60G60; 62M15, 53C65, 42C15

1 Introduction

1.1 Motivations and General Framework

In this paper, we shall be concerned with geometric functionals and excursion probabilities for some nonlinear transforms evaluated on Fourier components of spherical random fields. More precisely, let $\{T(x), x \in S^2\}$ denote a Gaussian, zero-mean isotropic spherical random field, i.e. for some probability space

*Research supported by the ERC Grants n. 277742 *Pascal*, "Probabilistic and Statistical Techniques for Cosmological Applications".

$(\Omega, \mathfrak{F}, P)$ the application $T(x, \omega) \rightarrow \mathbb{R}$ is $\{\mathfrak{F} \times \mathcal{B}(S^2)\}$ measurable, $\mathcal{B}(S^2)$ denoting the Borel σ -algebra on the sphere. It is well-known that the following representation holds, in the mean square sense (see for instance [26], [29], [28]):

$$T(x) = \sum_{\ell m} a_{\ell m} Y_{\ell m}(x) = \sum_{\ell} T_{\ell}(x) , \quad T_{\ell}(x) = \sum_{m=-\ell}^{\ell} a_{\ell m} Y_{\ell m}(x) . \quad (1)$$

where $\{Y_{\ell m}(\cdot)\}$ denotes the family of spherical harmonics, and $\{a_{\ell m}\}$ the array of random spherical harmonic coefficients, which satisfy $\mathbb{E} a_{\ell m} \bar{a}_{\ell' m'} = C_{\ell} \delta_{\ell}^{\ell'} \delta_m^{m'}$; here, δ_a^b is the Kronecker delta function, and the sequence $\{C_{\ell}\}$ represents the angular power spectrum of the field. As pointed out in [30], under isotropy the sequence C_{ℓ} necessarily satisfies $\sum_{\ell} \frac{(2\ell+1)}{4\pi} C_{\ell} = \mathbb{E} T^2 < \infty$ and the random field $T(x)$ is mean square continuous. Under the slightly stronger assumption $\sum_{\ell \geq L} (2\ell+1) C_{\ell} = O(\log^{-2} L)$, the field can be shown to be a.s. continuous, an assumption that we shall exploit heavily below.

Our attention will be focussed on the Fourier components $\{T_{\ell}(x)\}$, which represent random eigenfunctions of the spherical Laplacian:

$$\Delta_{S^2} T_{\ell} = -\ell(\ell+1) T_{\ell} , \quad \ell = 1, 2, \dots$$

A lot of recent work has been focussed on the characterization of geometric features for $\{T_{\ell}\}$, under Gaussianity assumptions; for instance [55], [56] studied the asymptotic behaviour of the nodal domains, proving an earlier conjecture by Berry on the variance of (functionals of) the zero sets of T_{ℓ} . In an earlier contribution, [12] had focussed on the *Defect* or signed area, i.e. the difference between the positive and negative regions; a Central Limit Theorem for these statistics and more general nonlinear transforms of Fourier components was recently established by [34]. These studies have been motivated, for instance, by the analysis of so-called Quantum Chaos (see again [12]), where the behaviour of random eigenfunctions is taken as an approximation for the asymptotics in deterministic case, under complex boundary conditions. More often, spherical eigenfunctions emerge naturally from the analysis of the Fourier components of spherical random fields, as in (1). In the latter circumstances, several functionals of T_{ℓ} assume a great practical importance: to mention a couple, the squared norm of T_{ℓ} provides an unbiased sample estimate for the angular power spectrum C_{ℓ} ,

$$\mathbb{E} \left\{ \int_{S^2} T_{\ell}^2(x) dx \right\} = (2\ell+1) C_{\ell} ,$$

while higher-order power lead to estimates of the so-called polyspectra, which have a great importance in the analysis of non-Gaussianity (see e.g. [29]).

The previous discussion shows that the analysis of nonlinear functionals of $\{T_{\ell}\}$ may have a great importance for statistical applications, especially in the framework of cosmological data analysis. In this area, a number of papers have searched for deviations of geometric functionals from the expected behaviour under Gaussianity. For instance, the so-called Minkowski functionals have been

widely used as tools to probe non-Gaussianity of the field $T(x)$, see [35] and the references therein. On the sphere, Minkowski functionals correspond to the area, the boundary length and the Euler-Poincaré characteristic of excursion sets, and up to constants they correspond to the Lipschitz-Killing Curvatures we shall consider in this paper, see [1], p.144. Many other works have also focussed on local deviations from the Gaussianity assumption, mainly exploiting the properties of integrated higher order moments (polyspectra), see [43], [44].

In general, the works aimed at the analysis of local phenomena are often based upon wavelets-like constructions, rather than standard Fourier analysis. The astrophysical literature on these issues is vast, see for instance [37], [45] and the references therein. Indeed, the double localization properties of wavelets (in real and harmonic domain) turn out usually to be extremely useful when handling real data.

In this paper, we shall focus on sequence of spherical random fields which can be viewed as averaged forms of the spherical eigenfunctions, e.g.

$$\beta_j(x) = \sum_{\ell} b\left(\frac{\ell}{B^j}\right) T_{\ell}(x), \quad j = 1, 2, 3 \dots$$

for $b(\cdot)$ a weight function whose properties we shall discuss immediately. The fields $\{\beta_j(x)\}$ can indeed be viewed as a representation of the coefficients from a continuous wavelet transform from $T(x)$, at scale j . More precisely, consider the kernel

$$\begin{aligned} \Psi_j(\langle x, y \rangle) &: = \sum_{\ell} b\left(\frac{\ell}{B^j}\right) \frac{2\ell+1}{4\pi} P_{\ell}(\langle x, y \rangle) \\ &= \sum_{\ell} b\left(\frac{\ell}{B^j}\right) \sum_{m=-\ell}^{\ell} Y_{\ell m}(x) \bar{Y}_{\ell m}(y). \end{aligned}$$

Assuming that $b(\cdot)$ is smooth (e.g. C^{∞}), compactly supported in $[B^{-1}, B]$, and satisfying the partition of unity property $\sum_j b^2(\frac{\ell}{B^j}) = 1$, for all $\ell > B$, where is a fixed "bandwidth" parameter s.t. $B > 1$. Then $\Psi_j(\langle x, y \rangle)$ can be viewed as a continuous version of the needlet transform, which was introduced by Narcowich et al. in [38], and considered from the point of view of statistics and cosmological data analysis by many subsequent authors, starting from [8], [31], [42]. In this framework, the following localization property is now well-known: for all $M \in \mathbb{N}$, there exists a constant C_M such that

$$|\Psi_j(\langle x, y \rangle)| \leq \frac{C_M B^{2j}}{\{1 + B^j d(x, y)\}^M},$$

where $d(x, y) = \arccos(\langle x, y \rangle)$ is the usual geodesic distance on the sphere. Heuristically, the field

$$\beta_j(x) = \int_{S^2} \Psi_j(\langle x, y \rangle) T(y) dy = \sum_{\ell} b\left(\frac{\ell}{B^j}\right) T_{\ell}(x)$$

is then only locally determined, i.e., for B^j large enough its value depends only from the behaviour of $T(y)$ in a neighbourhood of x . This is a very important property, for instance when dealing with spherical random fields which can only be partially observed, the canonical example being provided by the masking effect of the Milky Way on Cosmic Microwave Background (CMB) radiation.

It is hence very natural to produce out of $\{\beta_j(x)\}$ nonlinear statistics of great practical relevance. To provide a concrete example, a widely disputed theme in CMB data analysis concerns the existence of asymmetries in the angular power spectrum; it has been indeed often suggested that the angular power $\{C_\ell\}$ may exhibit different behaviour for different subsets of the sky, at least over some multipole range, see for instance [21], [43]. It is readily seen that

$$\mathbb{E} \{ \beta_j^2(x) \} = \sum_{\ell} b\left(\frac{\ell}{B^j}\right) \frac{2\ell+1}{4\pi} C_{\ell} ,$$

which hence suggests a natural “local” estimator for a binned form of the angular power spectrum. More precisely, it is natural to consider some form of averaging and introduce

$$g_{j;2}(z) := \int_{S^2} K(\langle z, x \rangle) \beta_j^2(x) dx .$$

For instance, should we consider the behaviour of the angular power spectrum on the northern and southern hemisphere, we might focus on

$$g_{j;2}(N) := \int_{S^2} K(\langle N, x \rangle) \beta_j^2(x) dx , \quad g_{j;2}(S) := \int_{S^2} K(\langle S, x \rangle) \beta_j^2(x) dx ,$$

where $K(\langle a, \cdot \rangle) := \mathbb{I}_{[0, \frac{\pi}{2}]}(\langle a, \cdot \rangle)$ is the indicator function of the hemisphere centred on $a \in S^2$, and N, S denote respectively the North and South Poles (compare [21], [43], [10] and the references therein). More generally, we shall be concerned with statistics of the form

$$g_{j;q}(z) := \int_{S^2} K(\langle z, x \rangle) H_q(\beta_j(x)) dx , \quad (2)$$

where $H_q(\cdot)$ is the Hermite polynomial of q -th order; for instance, for $q = 3$ these procedures can be exploited to investigate local variation in Gaussian and non-Gaussian features (see [44] and below for more discussion and details).

1.2 Main Result

The purpose of this paper is to study the asymptotic behaviour for the expected value of the Euler characteristic and other geometric functionals for the excursion regions of sequences of fields such as $\{g_{j;q}(\cdot)\}$, and to exploit these results to obtain excursion probabilities in nonGaussian circumstances. Indeed, on one hand these geometric functionals are of interest by themselves, as they provide the basis for implementing goodness-of-fit tests (compare [35]); on the other hand, they provide the clue for approximations of the excursion probabilities

for $\{g_{j;q}(\cdot)\}$, by means of some weak convergence results we shall establish, in combination with some now classical arguments described in detail in the monograph [1].

It is important to stress that our results are obtained under a setting which is quite different from usual. In particular, the asymptotic theory is investigated in the high frequency sense, e.g. assuming that a single realization of a spherical random field is observed at higher and higher resolution as more and more refined experiments are implemented. This is the setting adopted in [29], see also [5],[27],[53],[46] for the related framework of fixed-domain asymptotics.

Because of the nature of high-frequency asymptotics, we cannot expect the finite-dimensional distributions of the processes we focus on to converge. This will require a more general notion of weak convergence, as developed for instance by [15], [17]. By means of this, we shall indeed show how to evaluate asymptotically valid excursion probabilities, which provide a natural solution for hypothesis testing problems. Indeed, the main result of the paper, Theorem 21, provides a very explicit bound for the excursion probabilities of nonGaussian fields such as (2), e.g.

$$\limsup_{j \rightarrow \infty} \left| \Pr \left\{ \sup_{x \in S^2} \tilde{g}_{j;q}(x) > u \right\} - \{2(1 - \Phi(u)) + 2u\phi(u)\lambda_{j;q}\} \right| \leq \exp \left(-\frac{\alpha u^2}{2} \right), \quad (3)$$

where $\tilde{g}_{j;q}(x)$ has been normalized to have unit variance, $\phi(\cdot)$, $\Phi(\cdot)$ denote standard Gaussian density and distribution function, $\alpha > 1$ is some constant and the parameters $\lambda_{j;q}$ have analytic expressions in terms of generalized convolutions of angular power spectra, see (22), (18).

1.3 Plan of the paper

The plan of the paper is as follows: in Section 2 we review some background results on random fields and geometry, mainly referring to the now classical monograph [1]. Section 3 specializes these results to spherical random fields, for which some background theory is also provided, and provides some simple evaluations for Lipschitz-Killing curvatures related to excursion sets for harmonic components of such fields. More interesting Gaussian subordinated fields are considered in Section 4, where some detailed computations for covariances in general Gaussian subordinated circumstances are also provided. Section 5 provides the main convergence results, i.e. shows how the distribution of these random elements are asymptotically proximal (in the sense of [15]) to those of a Gaussian sequence with the same covariances. This result is then exploited in Section 6, to provide the proof of (3). A number of possible applications on real cosmological data sets are discussed throughout the paper.

2 Background: random fields and geometry

This section is devoted to recall basic integral geometric concepts, to state the Gaussian kinematic fundamental formula, and to discuss its application in

evaluating the excursion probabilities. This theory has been developed in a series of fundamental papers by R.J. Adler, J.E. Taylor and coauthors (see [48], [50], [49], [3]), and it is summarized in the monographs [1], [2] which are our main references in this Section (see also [6], [7] for a different approach, and [51], [13], [4] for some very recent developments in this area).

2.1 Lipschitz Killing curvatures and Gaussian Minkowski functionals

There are a number of ways to define Lipschitz-Killing curvatures, but perhaps the easiest is via the so-called tube formulae, which, in its original form is due to Hotelling [22] and Weyl [54]. To state the tube formula, let M be an m -dimensional smooth subset of \mathbb{R}^n such that ∂M is a C^2 manifold endowed with the canonical Riemannian structure on \mathbb{R}^n . The tube of radius ρ around M is defined as

$$\text{Tube}(M, \rho) = \{x \in \mathbb{R}^n : d(x, M) \leq \rho\}, \quad (4)$$

where,

$$d(x, M) = \inf_{y \in M} \|x - y\|. \quad (5)$$

Then according to Weyl's tube formula (see [1]), the Lebesgue volume of this constructed tube, for small enough ρ , is given by

$$\lambda_n(\text{Tube}(M, \rho)) = \sum_{j=0}^m \rho^{n-j} \omega_{n-j} \mathcal{L}_j(M), \quad (6)$$

where ω_j is the volume of the j -dimensional unit ball and $\mathcal{L}_j(M)$ is the j^{th} -Lipschitz-Killing curvature (LKC) of M . A little more analysis shows that $\mathcal{L}_m(M) = \mathcal{H}_m(M)$, the m -dimensional Hausdorff measure of M , and that $\mathcal{L}_0(M)$ is the Euler-Poincaré characteristic of M . Although the remaining LKCs have less transparent interpretations, it is easy to see that they satisfy simple scaling relationships, in that $\mathcal{L}_j(\alpha M) = \alpha^j \mathcal{L}_j(M)$ for all $1 \leq j \leq m$, where $\alpha M = \{x \in \mathbb{R}^n : x = \alpha y \text{ for some } y \in M\}$. Furthermore, despite the fact that defining the \mathcal{L}_j via (6) involves the embedding of M in \mathbb{R}^n , the $\mathcal{L}_j(M)$ are actually intrinsic, and so are independent of the ambient space.

Apart from their appearance in the tube formula (6), there are a number of other ways in which to define the LKCs. One such (non-intrinsic) way which signifies the dependence of the LKCs on the Riemannian metric is through the shape operator. Let M be an m -dimensional C^2 manifold embedded in \mathbb{R}^n ; then

$$\mathcal{L}_k(M) = K_{n,m,k} \int_M \int_{S(N_x M)} \text{Tr}(S_\nu^{(m-k)}) 1_{N_x M}(-\nu) \mathcal{H}_{n-m-1}(d\nu) \mathcal{H}_{m-1}(dx), \quad (7)$$

where, $K_{n,m,k} = \frac{1}{(m-k)!} \frac{\Gamma(\frac{(n-k)}{2})}{(2\pi)^{(n-k)/2}}$, and $S(N_x M)$ denotes a sphere in the normal space $N_x M$ of M at the point $x \in M$.

Closely related to the LKCs are set functionals called the Gaussian Minkowski functionals (GMFs), which are defined via a Gaussian tube formula. Consider the Gaussian measure, $\gamma_n(dx) = (2\pi)^{-n/2} e^{-\|x\|^2/2} dx$, instead of the standard Lebesgue measure in (6); the Gaussian tube formula is then given by

$$\gamma_n((M, \rho)) = \sum_{k \geq 0} \frac{\rho^k}{k!} \mathcal{M}_k^{\gamma_n}(M), \quad (8)$$

where the coefficients $\mathcal{M}_k^{\gamma_n}(M)$'s are the GMFs (for technical details, we refer the reader to [1]). We note that these set functionals, like their counterparts in (6) can be expressed as integrals over the manifold and its normal space (cf. [1]).

2.2 Excursion probabilities and the Gaussian kinematic fundamental formula

A classical problem in stochastic processes is to compute the excursion probability or the suprema probability

$$P\left(\sup_{x \in M} f(x) \geq u\right),$$

where, as before, f is a random field defined on the parameter space M . In the case when f happens to be a centered Gaussian field with constant variance σ^2 defined on M , a piecewise smooth manifold, then by the arguments set forth in Chapter 14 of [1], we have that

$$\left|P\left\{\sup_{x \in M} f(x) \leq u\right\} - \mathbb{E}\{\mathcal{L}_0(A_u(f; M))\}\right| < O\left(\exp\left(-\frac{\alpha u^2}{2\sigma^2}\right)\right), \quad (9)$$

where $\mathcal{L}_0(A_u(f; M))$ is, as defined earlier, the Euler-Poincaré characteristic of the excursion set $A_u(f; M) = \{x \in M : f(x) \geq u\}$, and $\alpha > 1$ is a constant, which depends on the field f and can be determined (see Theorem 14.3.3 of [1]).

At first sight, from (9) it may appear that we may have to deal with a hard task, e.g. that of evaluating $\mathbb{E}\{\mathcal{L}_0(A_u(f; M))\}$. This task, however, is greatly simplified due to the *Gaussian kinematic fundamental formula* (Gaussian-KFF) (see Theorems 15.9.4-15.9.5 in [1]), which states that, for a smooth $M \subset \mathbb{R}^N$

$$\begin{aligned} & \mathbb{E}(\mathcal{L}_i^f(A_u(f, M))) \\ &= \sum_{\ell=0}^{\dim(M)-i} \binom{i+\ell}{\ell} \frac{\Gamma(\frac{i}{2}+1) \Gamma(\frac{\ell}{2}+1)}{\Gamma(\frac{i+\ell}{2}+1)} (2\pi)^{-\ell/2} \mathcal{L}_{i+\ell}^f(M) \mathcal{M}_\ell^\gamma([u, \infty)), \end{aligned}$$

e.g., in the special case of the Euler characteristic ($i = 0$)

$$\mathbb{E}\left\{\mathcal{L}_0^f(A_u(f; M))\right\} = \sum_{j=0}^{\dim(M)} (2\pi)^{-j/2} \mathcal{L}_j^f(M) \mathcal{M}_j^\gamma([u, \infty)), \quad (10)$$

where $\mathcal{L}_j^f(M)$ is the j -th LKC of M with respect to the induced metric g^f given by

$$g_x^f(Y_x, Z_x) = \mathbb{E} \{Y f(x) \cdot Z f(x)\},$$

for $X_x, Y_x \in T_x M$, the tangent space at $x \in M$. The Gaussian kinematic fundamental formula will play a crucial role in all the developments to follow in the subsequent sections.

3 Spherical Gaussian fields

In this Section we shall start from some simple results on the evaluation of the expected values of Lipschitz-Killing curvatures for sequences of spherical Gaussian processes. These results will be rather straightforward applications of the Gaussian kinematic fundamental formula (10), and are collected here for completeness and as a bridge towards the more complicated case of nonlocal transforms of Gaussian subordinated processes, to be considered later.

Note first that for a unit variance Gaussian field on the sphere $f : S^2 \rightarrow \mathbb{R}$, the expected value of the Euler-Poincaré characteristic of the excursion set $A_u(f; S^2) = \{x \in S^2 : f(x) \geq u\}$ is given by

$$\mathbb{E} \{ \mathcal{L}_0(A_u(f, S^2)) \}$$

$$= \mathcal{L}_0^f(S^2) \mathcal{M}_0^\gamma([u, \infty)) + (2\pi)^{-1/2} \mathcal{L}_1^f(S^2) \mathcal{M}_1^\gamma([u, \infty)) + (2\pi)^{-1} \mathcal{L}_2^f(S^2) \mathcal{M}_2^\gamma([u, \infty)) ,$$

for

$$\mathcal{M}_0^\gamma([u, \infty)) = \int_u^\infty \phi(x) dx, \quad \mathcal{M}_j^\gamma([u, \infty)) = H_{j-1}(u) \phi(u),$$

where $\phi(\cdot)$ denotes the density of a real valued standard normal random variable, and $H_j(u)$ denotes the Hermite polynomials,

$$H_j(u) = (-1)^j (\phi(u))^{-1} \frac{d^j}{du^j} \phi(u) \text{ and } H_{-1}(u) = 1 - \Phi(u),$$

while $\mathcal{L}_k^f(S^2)$ are the usual Lipschitz-Killing curvatures, under the induced Gaussian metric, i.e.

$$\mathcal{L}_k^f(S^2) := \frac{(-2\pi)^{-(2-j)/2}}{2} \int_{S^2} Tr(R^{(N-k)/2}) Vol_{g^f} ;$$

here, R is the Riemannian curvature tensor and Vol_{g^f} is the volume form, under the induced Gaussian metric, given by

$$g^f(X, Y) := \mathbb{E} \{X f \cdot Y f\} = XY \mathbb{E}(f^2) .$$

We recall that $\mathcal{L}_0(M)$ is a topological invariant and does not depend on the metric; in particular, $\mathcal{L}_0(S^2) \equiv 2$. Moreover, because the sphere is an (even-) 2-dimensional manifold, $\mathcal{L}_1^f(S^2)$ is identically zero.

As mentioned before, we start from some very simple result on the Fourier components and wavelets transforms of Gaussian fields, e.g. the expected value of the Euler-Poincaré characteristic for two forms of harmonic components, namely

$$T_\ell(x) = \sum_{m=-\ell}^{\ell} a_{\ell m} Y_{\ell m}(x) \text{ and } \beta_j(x) = \sum_{\ell} b\left(\frac{\ell}{B^j}\right) T_\ell(x) ,$$

the first representing a Fourier component at the multipole ℓ , the second a field of continuous needlet/wavelet coefficients at scale j . We normalize these processes to unit variance by taking

$$\tilde{T}_\ell(x) = \frac{T_\ell(x)}{\sqrt{\frac{2\ell+1}{4\pi} C_\ell}} , \text{ and } \tilde{\beta}_j(x) = \frac{\beta_j(x)}{\sqrt{\sum_{\ell} b^2\left(\frac{\ell}{B^j}\right) \frac{(2\ell+1)}{4\pi} C_\ell}} .$$

Expressions for Lipschitz-Killing curvatures of excursion sets generated by Gaussian fields are basically known in the astrophysical community (see [35] and the references therein). Without too much attention on rigour, our results below could be derived from these expressions, but nevertheless we decided to report them for completeness and to provide complete mathematical proofs, based upon equation (10).

Lemma 1 *We have*

$$\begin{aligned} \tilde{\mathcal{L}}_2^{\tilde{\beta}_j}(S^2) &= 4\pi \frac{\sum_{\ell} b^2\left(\frac{\ell}{B^j}\right) (2\ell+1) C_\ell P'_\ell(1)}{\sum_{\ell} b^2\left(\frac{\ell}{B^j}\right) (2\ell+1) C_\ell} \\ &= 4\pi \frac{\sum_{\ell} b^2\left(\frac{\ell}{B^j}\right) (2\ell+1) C_\ell \frac{\ell(\ell+1)}{2}}{\sum_{\ell} b^2\left(\frac{\ell}{B^j}\right) (2\ell+1) C_\ell} . \end{aligned}$$

Proof. Recall first that, in standard spherical coordinates

$$P_\ell(\langle x, y \rangle) = P_\ell(\sin \vartheta_x \sin \vartheta_y \cos(\phi_x - \phi_y) + \cos \vartheta_x \cos \vartheta_y) .$$

Some simple algebra then yields

$$\left. \frac{\partial^2}{\partial \vartheta_x \partial \vartheta_y} P_\ell(\langle x, y \rangle) \right|_{x=y} = \frac{\partial^2}{\sin \vartheta_x \sin \vartheta_y \partial \phi_x \partial \phi_y} P_\ell(\langle x, y \rangle) \Big|_{x=y} = P'_\ell(1) ,$$

and

$$\left. \frac{\partial^2}{\sin \vartheta_x \partial \vartheta_y \partial \phi_x} P_\ell(\langle x, y \rangle) \right|_{x=y} = 0 .$$

The geometric meaning of the latter result is that the process is still isotropic under the new transformation, whence the derivatives along the two directions are still independent. We thus have that

$$\tilde{\mathcal{L}}_2^{\tilde{\beta}_j}(S^2) = \int_{S^2} \left\{ \det \begin{bmatrix} \frac{\partial^2}{\partial \vartheta_x \partial \vartheta_y} \Gamma(x, y) \Big|_{x=y} & \frac{\partial^2}{\sin \vartheta_x \partial \phi_x \partial \vartheta_y} \Gamma(x, y) \Big|_{x=y} \\ \frac{\partial^2}{\sin \vartheta_y \partial \phi_y \partial \vartheta_x} \Gamma(x, y) \Big|_{x=y} & \frac{\partial^2}{\sin \vartheta_x \sin \vartheta_y \partial \phi_x \partial \phi_y} \Gamma(x, y) \Big|_{x=y} \end{bmatrix} \right\}^{1/2} \sin \vartheta d\vartheta d\phi$$

$$\begin{aligned}
&= 4\pi \left\{ \det \begin{bmatrix} \frac{\sum_{\ell} b^2(\frac{\ell}{B^j}) C_{\ell} \frac{2\ell+1}{4\pi} P'_{\ell}(1)}{\sum_{\ell} b^2(\frac{\ell}{B^j}) \frac{(2\ell+1)}{4\pi} C_{\ell}} & 0 \\ 0 & \frac{\sum_{\ell} b^2(\frac{\ell}{B^j}) C_{\ell} \frac{2\ell+1}{4\pi} P'_{\ell}(1)}{\sum_{\ell} b^2(\frac{\ell}{B^j}) \frac{(2\ell+1)}{4\pi} C_{\ell}} \end{bmatrix} \right\}^{1/2} \\
&= 4\pi \frac{\sum_{\ell} b^2(\frac{\ell}{B^j}) \frac{(2\ell+1)}{4\pi} C_{\ell} P'_{\ell}(1)}{\sum_{\ell} b^2(\frac{\ell}{B^j}) \frac{(2\ell+1)}{4\pi} C_{\ell}} .
\end{aligned}$$

Now recall that $P'_{\ell}(1) = \frac{\ell(\ell+1)}{2}$, whence the claim is established. ■

Remark 2 Note that since the random field β_j is an isotropic Gaussian random field, the Lipschitz-Killing curvatures of S^2 under the metric induced by the field β_j are given by

$$\mathcal{L}_i^{\tilde{\beta}_j}(S^2) = \lambda_j^i \mathcal{L}_i(S^2),$$

where $\mathcal{L}_i(S^2)$ is the i -th LKC under the usual Euclidean metric, and λ_j is the second spectral moment of $\tilde{\beta}_j$ (cf. [1]). This result is true for all isotropic and unit variance Gaussian random fields.

The second auxiliary result that we shall need is Theorem 13.2.1 in [1], specialized to spherical random fields with unit variance.

Lemma 3 ([1]) For the Gaussian isotropic field $\tilde{\beta}_j : S^2 \rightarrow \mathbb{R}$, such that $\mathbb{E}\tilde{\beta}_j = 0$, $\mathbb{E}\tilde{\beta}_j^2 = 1$, $\tilde{\beta}_j \in C^2(S^2)$ almost surely, we have that

$$\mathbb{E} \left\{ \mathcal{L}_i(A_u(\tilde{\beta}_j(x), S^2)) \right\} = \sum_{\ell=0}^{\dim(S^2)-i} \begin{bmatrix} i+\ell \\ \ell \end{bmatrix} \lambda^{\ell/2} \rho_{\ell}(u) \mathcal{L}_{i+\ell}(S^2),$$

where

$$\begin{bmatrix} i+\ell \\ \ell \end{bmatrix} := \begin{pmatrix} i+\ell \\ \ell \end{pmatrix} \frac{\omega_{i+\ell}}{\omega_i \omega_{\ell}}, \quad \omega_i = \frac{\pi^{i/2}}{\Gamma(\frac{i}{2} + 1)},$$

$$\rho_{\ell}(u) = (2\pi)^{-\ell/2} \mathcal{M}_{\ell}^{\gamma}([u, \infty)) = (2\pi)^{-(\ell+1)/2} H_{\ell-1}(u) e^{-u^2/2},$$

so that

$$\rho_0(u) = (2\pi)^{-1/2} \sqrt{2\pi} (1 - \Phi(u)) e^{u^2/2} e^{-u^2/2} = (1 - \Phi(u)),$$

$$\rho_1(u) = \frac{1}{2\pi} e^{-u^2/2}, \quad \rho_2(u) = \frac{1}{\sqrt{(2\pi)^3}} u e^{-u^2/2}.$$

Here

$$\lambda = \mathbb{E}\beta_{j;\vartheta}^2 = \mathbb{E}\beta_{j;\phi}^2, \quad \beta_{j;\vartheta} = \frac{\partial}{\partial \vartheta} \beta_j(\vartheta, \phi), \quad \beta_{j;\phi} = \frac{\partial}{\sin \vartheta \partial \phi} \beta_j(\vartheta, \phi).$$

Proof. We start by recalling that

$$\mathbb{E} \left\{ \tilde{\beta}_{j;\vartheta}^2 \right\} = \frac{\partial^2}{\partial \vartheta^2} \mathbb{E} \left\{ \tilde{\beta}_j^2 \right\} = \frac{\sum_{\ell} b^2(\frac{\ell}{B^j}) C_{\ell} \frac{2\ell+1}{4\pi} P'_{\ell}(1)}{\sum_{\ell} b^2(\frac{\ell}{B^j}) C_{\ell} \frac{2\ell+1}{4\pi}} ,$$

whence

$$\begin{aligned} \mathbb{E} \left\{ \mathcal{L}_0(A_u(\tilde{\beta}_j(x), S^2)) \right\} &= \{1 - \Phi(u)\} \mathcal{L}_0(S^2) + \lambda \frac{ue^{-u^2/2}}{\sqrt{(2\pi)^3}} \mathcal{L}_2(S^2) \\ &= 2 \{1 - \Phi(u)\} + \lambda \frac{ue^{-u^2/2}}{\sqrt{(2\pi)^3}} 4\pi \\ &= 2 \{1 - \Phi(u)\} + 4\pi \left\{ \frac{\sum_{\ell} b^2(\frac{\ell}{B^j}) C_{\ell} \frac{2\ell+1}{4\pi} P'_{\ell}(1)}{\sum_{\ell} b^2(\frac{\ell}{B^j}) C_{\ell} \frac{2\ell+1}{4\pi}} \right\} \frac{ue^{-u^2/2}}{\sqrt{(2\pi)^3}} . \end{aligned}$$

Also

$$\begin{aligned} \mathbb{E} \left\{ \mathcal{L}_1(A_u(\tilde{\beta}_j(x), S^2)) \right\} &= \sum_{\ell=0}^1 \left[\begin{matrix} 1+\ell \\ \ell \end{matrix} \right] \lambda^{\ell/2} \rho_{\ell}(u) \mathcal{L}_{1+\ell}(S^2) = \left[\begin{matrix} 2 \\ 1 \end{matrix} \right] \lambda^{1/2} \rho_1(u) \mathcal{L}_2(S^2) \\ &= \frac{\pi}{2} \times 4\pi \times \left\{ \frac{\sum_{\ell} b^2(\frac{\ell}{B^j}) C_{\ell} \frac{2\ell+1}{4\pi} P'_{\ell}(1)}{\sum_{\ell} b^2(\frac{\ell}{B^j}) C_{\ell} \frac{2\ell+1}{4\pi}} \right\}^{1/2} \frac{1}{\sqrt{2\pi}} \frac{e^{-u^2/2}}{\sqrt{2\pi}} \\ &= \pi \left\{ \frac{\sum_{\ell} b^2(\frac{\ell}{B^j}) C_{\ell} \frac{2\ell+1}{4\pi} P'_{\ell}(1)}{\sum_{\ell} b^2(\frac{\ell}{B^j}) C_{\ell} \frac{2\ell+1}{4\pi}} \right\}^{1/2} e^{-u^2/2} . \end{aligned}$$

Finally

$$\mathbb{E} \left\{ \mathcal{L}_2(A_u(\tilde{\beta}_j(x), S^2)) \right\} = \rho_0(u) \mathcal{L}_2(S^2) = \{1 - \Phi(u)\} 4\pi .$$

Using the differential geometric definition of the Lipschitz-Killing curvatures, it is easy to observe that

$$2\mathbb{E} \left\{ \mathcal{L}_1(A_u(\tilde{T}_{\ell}(\cdot), S^2)) \right\} = \mathbb{E} \left\{ \text{len}(\partial A_u(T_{\ell}(\cdot), S^2)) \right\} ,$$

where $\text{len}(\partial A_u(\tilde{T}_{\ell}(\cdot), S^2))$ is the usual length of the boundary region of the excursion set, in the usual Hausdorff sense, which can also be expressed as $\mathcal{L}_1(\partial A_u(T_{\ell}(\cdot), S^2))$. In fact, writing $\partial A_u(f, S^2) = f^{-1}\{u\}$, we have

$$\mathbb{E} \left\{ \mathcal{L}_1^{\tilde{\beta}_j}(\partial A_u(T_{\ell}(\cdot), S^2)) \right\} = \sum_{k=0}^1 \left[\begin{matrix} k+1 \\ k \end{matrix} \right] (2\pi)^{-k/2} \mathcal{M}_k(\{u\}) \mathcal{L}_{k+1}^{\tilde{\beta}_j}(S^2) .$$

The Euler-Poincaré characteristic is zero here because we are dealing with a closed curve; hence we have

$$\begin{aligned} \mathbb{E} \left\{ \mathcal{L}_1^{\tilde{\beta}_j}(\partial A_u(T_{\ell}(\cdot), S^2)) \right\} &= \left[\begin{matrix} 2 \\ 1 \end{matrix} \right] (2\pi)^{-1/2} \left\{ 2 \frac{e^{-u^2}}{\sqrt{2\pi}} \right\} \lambda^{2/2} \mathcal{L}_2(S^2) \\ &= 2 \frac{\pi}{4} \frac{2}{2\pi} e^{-u^2/2} \left\{ \frac{\ell(\ell+1)}{2} \right\} 4\pi = 2\pi \left\{ \frac{\ell(\ell+1)}{2} \right\} e^{-u^2/2} , \end{aligned}$$

where we have used $\mathcal{M}_1(\{u\}) = 2 \frac{e^{-u^2/2}}{\sqrt{2\pi}}$. Hence

$$\begin{aligned} \mathbb{E} \left\{ \mathcal{L}_1(\partial A_u(\tilde{T}_\ell(\cdot), S^2)) \right\} &= \lambda^{-1/2} \mathbb{E} \left\{ \mathcal{L}_1^{\tilde{T}_\ell}(\partial A_u(\tilde{T}_\ell(\cdot), S^2)) \right\} \\ &= \lambda^{-1/2} 2\pi \left\{ \frac{\ell(\ell+1)}{2} \right\} e^{-u^2/2} = 2\pi \left\{ \frac{\ell(\ell+1)}{2} \right\}^{1/2} e^{-u^2/2} = \mathbb{E} \left\{ \text{len}(\partial A_u(\tilde{T}_\ell(\cdot), S^2)) \right\}, \end{aligned}$$

which for $u = 0$ fits with well-known results on the expected value of nodal lines for random spherical eigenfunctions (see [56] and the references therein). Likewise

$$\mathbb{E} \left\{ \text{len}(\partial A_u(\tilde{\beta}_j(\cdot), S^2)) \right\} = 2\pi \left\{ \frac{\sum_\ell b^2(\frac{\ell}{B^j}) C_\ell \frac{2\ell+1}{4\pi} P'_\ell(1)}{\sum_\ell b^2(\frac{\ell}{B^j}) C_\ell \frac{2\ell+1}{4\pi}} \right\}^{1/2} e^{-u^2/2}. \quad (11)$$

■

Remark 4 Note that, in this setting

$$\begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} = 1, \quad \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \frac{\pi}{2}.$$

In the case of spherical eigenfunctions, the previous Lemma takes the following simpler form; the proof is entirely analogous, and hence omitted.

Corollary 5 For the field $\{T_\ell(\cdot)\}$, we have that

$$\mathbb{E} \left\{ \mathcal{L}_0(A_u(\tilde{T}_\ell(\cdot), S^2)) \right\} = 2 \{1 - \Phi(u)\} + \frac{\ell(\ell+1)}{2} \frac{ue^{-u^2/2}}{\sqrt{(2\pi)^3}} 4\pi, \quad (12)$$

$$\mathbb{E} \left\{ \mathcal{L}_1(A_u(\tilde{T}_\ell(\cdot), S^2)) \right\} = \pi \left\{ \frac{\ell(\ell+1)}{2} \right\}^{1/2} e^{-u^2/2},$$

and

$$\mathbb{E} \left\{ \mathcal{L}_2(A_u(\tilde{T}_\ell(\cdot), S^2)) \right\} = 4\pi \times \{1 - \Phi(u)\}.$$

Remark 6 The expression (11) can be viewed as a weighted average of $\mathbb{E} \left\{ \text{len}(\partial A_u(T_\ell(\cdot), S^2)) \right\}$, with weights provided by $w_\ell := b^2(\frac{\ell}{B^j}) C_\ell \frac{2\ell+1}{4\pi}$. The sequence w_ℓ is summable; in a heuristic sense, we can argue that it must hence be eventually decreasing; we thus expect that, for ℓ large enough

$$\frac{\sum_\ell b^2(\frac{\ell}{B^j}) C_\ell \frac{2\ell+1}{4\pi} P'_\ell(1)}{\sum_\ell b^2(\frac{\ell}{B^j}) C_\ell \frac{2\ell+1}{4\pi}} \ll \frac{\ell(\ell+1)}{2},$$

and hence

$$\mathbb{E} \left\{ \text{len}(\partial A_u(\tilde{\beta}_j(\cdot), S^2)) \right\} \ll \mathbb{E} \left\{ \text{len}(\partial A_u(\tilde{T}_\ell(\cdot), S^2)) \right\}.$$

This fits with the heuristic understanding that the sequence of fields $\{\beta_j(\cdot)\}$, representing an average, is smoother than the subordinating sequence $\{T_\ell(\cdot)\}$; the latter is thus expected to have rougher excursion sets, and hence greater boundary regions. This heuristic argument can be made rigorous imposing some regularity conditions on the behaviour of the angular power spectrum $\{C_\ell\}$.

Much more explicit results can of course be obtained by setting a more specific form for the behaviour of the angular power spectrum $\{C_\ell\}$ and the weighting kernel $b(\cdot)$; for instance, in a CMB related environment it is natural to consider the *Sachs-Wolfe* angular power spectrum $C_\ell \sim G\ell^{-\alpha}$, some $G > 0$, $\alpha > 2$, (see [18]).

4 Gaussian subordinated fields

4.1 Local Transforms of $\beta_j(\cdot)$

For statistical applications, it is often more interesting to consider nonlinear transforms of random fields. For instance, in a CMB related environment a lot of efforts have been spent to investigate local fluctuations of angular power spectra; to this aim, moving averages of squared wavelet/needlet coefficients are usually computed, see for instance [43] and the references therein. Our purpose here is to derive some rigorous results on the behaviour of these statistics.

To this aim, let us consider first the simple squared field

$$H_{2j}(x) := H_2(\tilde{\beta}_j(x)) = \frac{\beta_j^2(x)}{\mathbb{E}\beta_j^2(x)} - 1.$$

The expected value of Lipschitz-Killing curvatures for the excursion regions of such fields is easily derived, indeed by the general Gaussian kinematic formula we have

$$\begin{aligned} \mathbb{E} \left\{ \mathcal{L}_0^{\tilde{\beta}_j}(A_u(H_2; S^2)) \right\} &= \sum_{k=0}^2 (2\pi)^{-k/2} \mathcal{L}_k^{\tilde{\beta}_j}(S^2) \mathcal{M}_k^{\mathcal{N}}((-\infty, -\sqrt{u+1}) \cup (\sqrt{u+1}, \infty)) \\ &= \sum_{k=0}^2 (2\pi)^{-k/2} \mathcal{L}_k^{\tilde{\beta}_j}(S^2) 2\mathcal{M}_k^{\mathcal{N}}((\sqrt{u+1}, \infty)) \\ &= 4(1 - \Phi(\sqrt{u+1})) + \frac{1}{2\pi} \frac{\sum_\ell b^2(\frac{\ell}{B^j}) \frac{2\ell+1}{4\pi} C_\ell P'_\ell(1)}{\sum_\ell b^2(\frac{\ell}{B^j}) \frac{2\ell+1}{4\pi} C_\ell} \mathcal{L}_2(S^2) \frac{e^{-(u+1)/2}}{\sqrt{2\pi}} 2\sqrt{u+1}. \end{aligned}$$

Likewise

$$\begin{aligned} \mathbb{E} \left\{ \mathcal{L}_1^{\tilde{\beta}_j}(A_u(H_2; S^2)) \right\} &= \sum_{k=0}^1 (2\pi)^{-k/2} \left[\begin{matrix} k+1 \\ k \end{matrix} \right] \mathcal{L}_{k+1}^{\tilde{\beta}_j}(S^2) \mathcal{M}_k^{\mathcal{N}}((-\infty, -\sqrt{u+1}) \cup (\sqrt{u+1}, \infty)) \\ &= \mathcal{L}_1^{\tilde{\beta}_j}(S^2) \mathcal{M}_0^{\mathcal{N}}((-\infty, -\sqrt{u+1}) \cup (\sqrt{u+1}, \infty)) \\ &\quad + (2\pi)^{-1/2} \frac{\pi}{2} \mathcal{L}_2^{\tilde{\beta}_j}(S^2) \mathcal{M}_1^{\mathcal{N}}((-\infty, -\sqrt{u+1}) \cup (\sqrt{u+1}, \infty)) \\ &= (2\pi)^{-1/2} \frac{\pi}{2} (4\pi \times \frac{\sum_\ell b^2(\frac{\ell}{B^j}) \frac{2\ell+1}{4\pi} C_\ell P'_\ell(1)}{\sum_\ell b^2(\frac{\ell}{B^j}) \frac{2\ell+1}{4\pi} C_\ell}) 2 \frac{e^{-(u+1)/2}}{\sqrt{2\pi}} \\ &= 2\pi \left(\frac{\sum_\ell b^2(\frac{\ell}{B^j}) \frac{2\ell+1}{4\pi} C_\ell P'_\ell(1)}{\sum_\ell b^2(\frac{\ell}{B^j}) \frac{2\ell+1}{4\pi} C_\ell} \right) e^{-(u+1)/2}, \end{aligned}$$

which implies for the Euclidean LKC

$$\mathbb{E} \left\{ \mathcal{L}_1(A_u(H_2; S^2)) \right\} = 2\pi \left\{ \frac{\sum_{\ell} b^2(\frac{\ell}{B^j}) \frac{2\ell+1}{4\pi} C_{\ell} P'_{\ell}(1)}{\sum_{\ell} b^2(\frac{\ell}{B^j}) \frac{2\ell+1}{4\pi} C_{\ell}} \right\}^{1/2} e^{-(u+1)/2},$$

and therefore

$$\mathbb{E} \left\{ \mathcal{L}_1(\partial A_u(H_2; S^2)) \right\} = 4\pi \left\{ \frac{\sum_{\ell} b^2(\frac{\ell}{B^j}) \frac{2\ell+1}{4\pi} C_{\ell} P'_{\ell}(1)}{\sum_{\ell} b^2(\frac{\ell}{B^j}) \frac{2\ell+1}{4\pi} C_{\ell}} \right\}^{1/2} e^{-(u+1)/2}.$$

Finally

$$\begin{aligned} \mathbb{E} \left\{ \tilde{\mathcal{L}}_2^{\tilde{\beta}_j}(A_u(H_2; S^2)) \right\} &= \mathcal{L}_2^{\tilde{\beta}_j}(S^2) \mathcal{M}_0^{\mathcal{N}}((-\infty, -\sqrt{u+1}) \cup (\sqrt{u+1}, \infty)) \\ &= 4\pi \left\{ \frac{\sum_{\ell} b^2(\frac{\ell}{B^j}) \frac{2\ell+1}{4\pi} C_{\ell} P'_{\ell}(1)}{\sum_{\ell} b^2(\frac{\ell}{B^j}) \frac{2\ell+1}{4\pi} C_{\ell}} \right\} 2(1 - \Phi(\sqrt{u+1})) \end{aligned}$$

entailing a Euclidean LKC

$$\mathbb{E} \left\{ \mathcal{L}_2(A_u(H_2; S^2)) \right\} = 4\pi \times 2(1 - \Phi(\sqrt{u+1})) .$$

It should be noted that the tail decay for the Euler characteristic and the boundary length is much slower than in the Gaussian case. This is consistent with the elementary fact that polynomial transforms shift angular power spectra at higher frequencies, hence yielding a rougher path behaviour. Likewise, for cubic transforms we have

$$\begin{aligned} \mathbb{E} \left\{ \tilde{\mathcal{L}}_0^{\tilde{\beta}_j}(A_u(\tilde{\beta}_j^3(x); S^2)) \right\} &= 2(1 - \Phi(\sqrt[3]{u})) + \frac{1}{2\pi} \frac{\sum_{\ell} b^2(\frac{\ell}{B^j}) \frac{2\ell+1}{4\pi} C_{\ell} P'_{\ell}(1)}{\sum_{\ell} b^2(\frac{\ell}{B^j}) \frac{2\ell+1}{4\pi} C_{\ell}} \mathcal{L}_2(S^2) \frac{e^{-(\sqrt[3]{u})^2/2}}{\sqrt{2\pi}} \sqrt[3]{u} , \\ \mathbb{E} \left\{ \tilde{\mathcal{L}}_1^{\tilde{\beta}_j}(A_u(\tilde{\beta}_j^3(x); S^2)) \right\} &= \pi \left(\frac{\sum_{\ell} b^2(\frac{\ell}{B^j}) \frac{2\ell+1}{4\pi} C_{\ell} P'_{\ell}(1)}{\sum_{\ell} b^2(\frac{\ell}{B^j}) \frac{2\ell+1}{4\pi} C_{\ell}} \right) e^{-(\sqrt[3]{u})^2/2}, \\ \mathbb{E} \left\{ \mathcal{L}_1(\partial A_u(\tilde{\beta}_j^3(x); S^2)) \right\} &= 2\pi \left\{ \frac{\sum_{\ell} b^2(\frac{\ell}{B^j}) \frac{2\ell+1}{4\pi} C_{\ell} P'_{\ell}(1)}{\sum_{\ell} b^2(\frac{\ell}{B^j}) \frac{2\ell+1}{4\pi} C_{\ell}} \right\}^{1/2} e^{-(\sqrt[3]{u})^2/2}, \end{aligned}$$

and finally

$$\mathbb{E} \left\{ \tilde{\mathcal{L}}_2^{\tilde{\beta}_j}(A_u(\tilde{\beta}_j^3(x); S^2)) \right\} = 4\pi \left\{ \frac{\sum_{\ell} b^2(\frac{\ell}{B^j}) \frac{2\ell+1}{4\pi} C_{\ell} P'_{\ell}(1)}{\sum_{\ell} b^2(\frac{\ell}{B^j}) \frac{2\ell+1}{4\pi} C_{\ell}} \right\} 2(1 - \Phi(\sqrt[3]{u}))$$

entailing a Euclidean LKC

$$\mathbb{E} \left\{ \mathcal{L}_2(A_u(\tilde{\beta}_j^3(x); S^2)) \right\} = 4\pi(1 - \Phi(\sqrt[3]{u})) .$$

Similar results could be easily derived for higher order polynomial transforms. However, although such findings may be useful for applications, as motivated above we believe it is much more important to focus on transforms that entail some form of local averaging, as these are likely to be more relevant for practitioners. To this issue we devote the rest of this section and a large part of the paper.

4.2 Nonlocal Transforms of $\beta_j(\cdot)$

We now consider the case of smoothed nonlinear functionals. We are interested, for instance, in studying the LKCs for local estimates of the angular power spectrum, which as mentioned before have already found many important applications in a CMB related framework. To this aim, we introduce, for every $x \in S^2$

$$g_{j;q}(x) := \int_{S^2} K(\langle x, y \rangle) H_q(\tilde{\beta}_j(y)) dy ; \quad (13)$$

throughout the sequel, we shall assume the kernel $K : [0, 1] \rightarrow \mathbb{R}$ to be rotational invariant and smooth, e.g. we assume that the following finite-order expansion holds:

$$K(\langle x, y \rangle) = \sum_{\ell}^{L_K} \frac{2\ell + 1}{4\pi} \kappa(\ell) P_{\ell}(\langle x, y \rangle) , \text{ some } L_K \in \mathbb{N} .$$

Here, as before we write $H_q(\cdot)$ for the Hermite polynomials. For $q = 1$, we just get the smoothed Gaussian process

$$g_j(x) := g_{j;1}(x) = \int_{S^2} K(\langle x, y \rangle) \tilde{\beta}_j(y) dy .$$

The practical importance of the analysis of fields such as $g_{j;q}(\cdot)$ can be motivated as follows. A crucial topic when dealing with cosmological data is the analysis of isotropy properties. For instance, in a CMB related framework a large amount of work has focussed on the possible existence of asymmetries in the behaviour of angular power spectra or bispectra across different hemispheres (see for instance [43], [44]). In these papers, powers of wavelet coefficients at some frequencies j are averaged over different hemispheres to investigate the existence of asymmetries/anisotropies in the CMB distribution; some evidence has been reported, for instance, for power asymmetries with respect to the Milky Way plane for frequencies corresponding to angular scales of a few degrees (such effects are related in the Cosmological literature to widely debated anomalies known as *the Cold Spot* and *the Axis of Evil*, see [10] and the references therein). To investigate these anomalies, statistics which can be viewed as discretized versions of $\sup_{x \in S^2} g_{j;q}(x)$ have been evaluated; their significance is typically tested against Monte Carlo simulations, under the null of isotropy. Our results below will provide the first rigorous derivation of asymptotic properties in this settings.

Our first Lemma is an immediate application of spherical Fourier analysis techniques. For notational simplicity and without loss of generality, we take until the end of this Section $\mathbb{E}\beta_j^2(x) = 1$, so we need no longer distinguish between β_j and $\tilde{\beta}_j$.

Lemma 7 *The field $g_j(x)$ is zero-mean, finite variance and isotropic, with covariance function*

$$\mathbb{E}\{g_j(x_1)g_j(x_2)\} = \sum_{\ell} b^2\left(\frac{\ell}{B^j}\right) \kappa^2(\ell) \frac{2\ell + 1}{4\pi} C_{\ell} P_{\ell}(\langle x_1, x_2 \rangle) .$$

Proof. Note first that

$$\begin{aligned}
\mathbb{E} \{g_j(x_1)g_j(x_2)\} &= \mathbb{E} \left\{ \int_{S^2} K(\langle x_1, y_1 \rangle) \beta_j(y_1) dy_1 \int_{S^2} K(\langle x_2, y_2 \rangle) \beta_j(y_2) dy_2 \right\} \\
&= \left\{ \int_{S^2 \times S^2} K(\langle x_1, y_1 \rangle) K(\langle x_2, y_2 \rangle) \mathbb{E} \{ \beta_j(y_1) \beta_j(y_2) \} dy_1 dy_2 \right\} \\
&= \int_{S^2 \times S^2} K(\langle x_1, y_1 \rangle) K(\langle x_2, y_2 \rangle) \sum_{\ell} b^2\left(\frac{\ell}{B^j}\right) \frac{2\ell+1}{4\pi} C_{\ell} P_{\ell}(\langle y_1, y_2 \rangle) .
\end{aligned}$$

Recall the reproducing kernel formula

$$\begin{aligned}
\int_{S^2} P_{\ell}(\langle x_1, y_1 \rangle) P_{\ell}(\langle y_1, y_2 \rangle) dy_1 &= \frac{4\pi}{2\ell+1} P_{\ell}(\langle x_1, y_2 \rangle) , \\
\int_{S^2} P_{\ell_1}(\langle x_1, y_1 \rangle) P_{\ell_2}(\langle y_1, y_2 \rangle) dy_1 &= 0 , \ell_1 \neq \ell_2 ,
\end{aligned}$$

whence

$$\begin{aligned}
&\int_{S^2 \times S^2} K(\langle x_1, y_1 \rangle) K(\langle x_2, y_2 \rangle) \sum_{\ell} b^2\left(\frac{\ell}{B^j}\right) \frac{2\ell+1}{4\pi} C_{\ell} P_{\ell}(\langle y_1, y_2 \rangle) \\
&= \int_{S^2 \times S^2} \sum_{\ell_1} \frac{2\ell_1+1}{4\pi} \kappa(\ell_1) P_{\ell_1}(\langle x_1, y_1 \rangle) \sum_{\ell_2} \frac{2\ell_2+1}{4\pi} \kappa(\ell_2) P_{\ell_2}(\langle x_2, y_2 \rangle) \\
&\quad \times \sum_{\ell} b^2\left(\frac{\ell}{B^j}\right) \frac{2\ell+1}{4\pi} C_{\ell} P_{\ell}(\langle y_1, y_2 \rangle) dy_1 dy_2 \\
&= \int_{S^2} \sum_{\ell} b^2\left(\frac{\ell}{B^j}\right) \frac{2\ell+1}{4\pi} C_{\ell} \sum_{\ell_1} \kappa(\ell_1) \sum_{\ell_2} \frac{2\ell_2+1}{4\pi} \kappa(\ell_2) P_{\ell_2}(\langle x_2, y_2 \rangle) \\
&\quad \times \int_{S^2} \frac{2\ell_1+1}{4\pi} P_{\ell_1}(\langle x_1, y_1 \rangle) P_{\ell}(\langle y_1, y_2 \rangle) dy_1 dy_2 \\
&= \sum_{\ell} b^2\left(\frac{\ell}{B^j}\right) \kappa(\ell) \frac{2\ell+1}{4\pi} C_{\ell} \sum_{\ell_2} \frac{2\ell_2+1}{4\pi} \kappa(\ell_2) \int_{S^2} P_{\ell_2}(\langle x_2, y_2 \rangle) P_{\ell}(\langle x_1, y_2 \rangle) dy_2 \\
&= \sum_{\ell} b^2\left(\frac{\ell}{B^j}\right) \kappa^2(\ell) \frac{2\ell+1}{4\pi} C_{\ell} P_{\ell}(\langle x_1, x_2 \rangle) ,
\end{aligned}$$

as claimed. ■

The derivation of analogous results in the case of $q \geq 2$ requires more work and extra notation. In particular, we shall need the Wigner's $3j$ coefficients,

which are defined by (for $m_1 + m_2 + m_3 = 0$, see [52], expression (8.2.1.5))

$$\begin{aligned} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \end{pmatrix} &:= (-1)^{\ell_1+m_1} \sqrt{2\ell_3+1} \left[\frac{(\ell_1+\ell_2-\ell_3)!(\ell_1-\ell_2+\ell_3)!(\ell_1-\ell_2+\ell_3)!}{(\ell_1+\ell_2+\ell_3+1)!} \right]^{1/2} \\ &\times \left[\frac{(\ell_3+m_3)!(\ell_3-m_3)!}{(\ell_1+m_1)!(\ell_1-m_1)!(\ell_2+m_2)!(\ell_2-m_2)!} \right]^{1/2} \\ &\times \sum_z \frac{(-1)^z (\ell_2+\ell_3+m_1-z)!(\ell_1-m_1+z)!}{z!(\ell_2+\ell_3-\ell_1-z)!(\ell_3+m_3-z)!(\ell_1-\ell_2-m_3+z)!}, \end{aligned}$$

where the summation runs over all z 's such that the factorials are non-negative. This expression becomes much neater for $m_1 = m_2 = m_3 = 0$, where we have

$$\begin{aligned} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ 0 & 0 & 0 \end{pmatrix} &= \\ \begin{cases} 0, & \text{for } \ell_1 + \ell_2 + \ell_3 \text{ odd} \\ (-1)^{\frac{\ell_1+\ell_2-\ell_3}{2}} \frac{[(\ell_1+\ell_2+\ell_3)/2]!}{[(\ell_1+\ell_2-\ell_3)/2]![(\ell_1-\ell_2+\ell_3)/2]![(-\ell_1+\ell_2+\ell_3)/2]!} \left\{ \frac{(\ell_1+\ell_2-\ell_3)!(\ell_1-\ell_2+\ell_3)!(-\ell_1+\ell_2+\ell_3)!}{(\ell_1+\ell_2+\ell_3+1)!} \right\}^{1/2} & \text{for } \ell_1 + \ell_2 + \ell_3 \text{ even} \end{cases} \end{aligned} \quad (14)$$

It is occasionally more convenient to focus on Clebsch-Gordan coefficients, which are related to the Wigner's by a simple change of normalization, e.g.

$$C_{\ell_1 m_1 \ell_2 m_2}^{\ell_3 m_3} := \frac{(-1)^{\ell_3-m_3}}{\sqrt{2\ell_3+1}} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & -m_3 \end{pmatrix}. \quad (15)$$

Wigner's $3j$ coefficients are elements of unitary matrices which intertwine alternative reducible representations of the group of rotations $SO(3)$, and because of this emerge naturally in the evaluation of multiple integrals of spherical harmonics, see for instance [29], Section 3.5.2. As a consequence, they also appear in the covariances of nonlinear transforms; for $q = 2$, we have indeed

Lemma 8 *The field $g_{j;2}(x)$ is zero-mean, finite variance and isotropic, with covariance function*

$$\begin{aligned} \mathbb{E} \{g_{j;2}(x_1)g_{j;2}(x_2)\} &= \\ 2 \sum_{\ell} \kappa^2(\ell) \frac{2\ell+1}{4\pi} \sum_{\ell_1 \ell_2} b^2\left(\frac{\ell_1}{B^j}\right) b^2\left(\frac{\ell_2}{B^j}\right) \frac{(2\ell_1+1)(2\ell_2+1)}{4\pi} C_{\ell_1} C_{\ell_2} \begin{pmatrix} \ell & \ell_1 & \ell_2 \\ 0 & 0 & 0 \end{pmatrix}^2 P_{\ell}(\langle x_1, x_2 \rangle). \end{aligned}$$

Proof. Note first that

$$\begin{aligned} \mathbb{E} \{g_{j;2}(x_1)g_{j;2}(x_2)\} &= \mathbb{E} \left\{ \int_{S^2} K(\langle x_1, y_1 \rangle) H_2(\beta_j(y_1)) dy_1 \int_{S^2} K(\langle x_2, y_2 \rangle) H_2(\beta_j(y_2)) dy_2 \right\} \\ &= \int_{S^2 \times S^2} K(\langle x_1, y_1 \rangle) K(\langle x_2, y_2 \rangle) \mathbb{E} \{H_2(\beta_j(y_1)) H_2(\beta_j(y_2))\} dy_1 dy_2 \end{aligned}$$

$$\begin{aligned}
&= 2 \int_{S^2 \times S^2} K(\langle x_1, y_1 \rangle) K(\langle x_2, y_2 \rangle) \left\{ \sum_{\ell} b^2 \left(\frac{\ell}{B^j} \right) \frac{2\ell+1}{4\pi} C_{\ell} P_{\ell}(\langle y_1, y_2 \rangle) \right\}^2 dy_1 dy_2 \\
&= 2 \int_{S^2 \times S^2} \sum_{\ell_1} \frac{2\ell_1+1}{4\pi} \kappa(\ell_1) P_{\ell_1}(\langle x_1, y_1 \rangle) \sum_{\ell_2} \frac{2\ell_2+1}{4\pi} \kappa(\ell_2) P_{\ell_2}(\langle x_2, y_2 \rangle) \\
&\quad \times \sum_{\ell_3 \ell_4} b^2 \left(\frac{\ell_3}{B^j} \right) b^2 \left(\frac{\ell_4}{B^j} \right) \frac{2\ell_3+1}{4\pi} \frac{2\ell_4+1}{4\pi} C_{\ell_3} C_{\ell_4} P_{\ell_3}(\langle y_1, y_2 \rangle) P_{\ell_4}(\langle y_1, y_2 \rangle) dy_1 dy_2 .
\end{aligned}$$

Now recall that

$$\begin{aligned}
&\int_{S^2} P_{\ell_1}(\langle x_1, y_1 \rangle) P_{\ell_3}(\langle y_1, y_2 \rangle) P_{\ell_4}(\langle y_1, y_2 \rangle) dy_1 \\
&= \frac{(4\pi)^3}{(2\ell_1+1)(2\ell_3+1)(2\ell_4+1)} \\
&\times \int_{S^2} \sum_{m_1 m_2 m_3} Y_{\ell_1 m_1}(y_1) \bar{Y}_{\ell_1 m_1}(x_1) Y_{\ell_3 m_3}(y_1) \bar{Y}_{\ell_3 m_3}(y_2) Y_{\ell_4 m_4}(y_1) \bar{Y}_{\ell_4 m_4}(y_2) dy_1 \\
&= \left(\frac{(4\pi)^5}{(2\ell_1+1)(2\ell_3+1)(2\ell_4+1)} \right)^{1/2} \\
&\times \sum_{m_1 m_3 m_4} \begin{pmatrix} \ell_1 & \ell_3 & \ell_4 \\ m_1 & m_3 & m_4 \end{pmatrix} \begin{pmatrix} \ell_1 & \ell_3 & \ell_4 \\ 0 & 0 & 0 \end{pmatrix} \bar{Y}_{\ell_1 m_1}(x_1) \bar{Y}_{\ell_3 m_3}(y_2) \bar{Y}_{\ell_4 m_4}(y_2) ,
\end{aligned}$$

Likewise

$$\begin{aligned}
&\int_{S^2} P_{\ell_2}(\langle x_2, y_2 \rangle) \bar{Y}_{\ell_3 m_3}(y_2) \bar{Y}_{\ell_4 m_4}(y_2) dy_2 \\
&= \frac{4\pi}{2\ell_2+1} \int_{S^2} \sum_{m_2} \bar{Y}_{\ell_2 m_2}(y_2) Y_{\ell_2 m_2}(x_2) \bar{Y}_{\ell_3 m_3}(y_2) \bar{Y}_{\ell_4 m_4}(y_2) dy_2 \\
&= \sqrt{\frac{(4\pi)(2\ell_3+1)(2\ell_4+1)}{2\ell_2+1}} \sum_{m_2} \begin{pmatrix} \ell_2 & \ell_3 & \ell_4 \\ m_2 & m_3 & m_4 \end{pmatrix} \begin{pmatrix} \ell_2 & \ell_3 & \ell_4 \\ 0 & 0 & 0 \end{pmatrix} Y_{\ell_2 m_2}(x_2) .
\end{aligned}$$

Using the orthonormality properties of Wigner's $3j$ coefficients (see again [29], Chapter 3.5), we have

$$\sum_{m_3 m_4} \begin{pmatrix} \ell_1 & \ell_3 & \ell_4 \\ m_1 & m_3 & m_4 \end{pmatrix} \begin{pmatrix} \ell_2 & \ell_3 & \ell_4 \\ m_2 & m_3 & m_4 \end{pmatrix} = \frac{\delta_{m_1}^{m_2} \delta_{\ell_1}^{\ell_2}}{(2\ell_1+1)} ,$$

whence we get

$$\begin{aligned}
&\mathbb{E} \{ g_{j;2}(x_1) g_{j;2}(x_2) \} = \\
&2 \sum_{\ell} \kappa^2(\ell) \frac{2\ell+1}{4\pi} \sum_{\ell_1 \ell_2} b^2 \left(\frac{\ell_1}{B^j} \right) b^2 \left(\frac{\ell_2}{B^j} \right) \frac{(2\ell_1+1)(2\ell_2+1)}{4\pi} C_{\ell_1} C_{\ell_2} \begin{pmatrix} \ell & \ell_1 & \ell_2 \\ 0 & 0 & 0 \end{pmatrix}^2 P_{\ell}(\langle x_1, x_2 \rangle) ,
\end{aligned}$$

as claimed. As a special case, the variance is provided by

$$\mathbb{E}g_{j,2}^2(x) = 2 \sum_{\ell} \kappa^2(\ell) \frac{2\ell+1}{4\pi} \sum_{\ell_1 \ell_2} b^2\left(\frac{\ell_1}{B^j}\right) b^2\left(\frac{\ell_2}{B^j}\right) \frac{(2\ell_1+1)(2\ell_2+1)}{4\pi} C_{\ell_1} C_{\ell_2} \begin{pmatrix} \ell & \ell_1 & \ell_2 \\ 0 & 0 & 0 \end{pmatrix}^2.$$

■

Remark 9 *Because the field $\{g_{j,2}(\cdot)\}$ has finite-variance and it is isotropic, it admits itself a spectral representation. Indeed, it is a simple computation to show that the corresponding angular power spectrum is provided by*

$$C_{\ell,j,2} := 2\kappa^2(\ell) \sum_{\ell_1 \ell_2} b^2\left(\frac{\ell_1}{B^j}\right) b^2\left(\frac{\ell_2}{B^j}\right) \frac{(2\ell_1+1)(2\ell_2+1)}{4\pi} C_{\ell_1} C_{\ell_2} \begin{pmatrix} \ell & \ell_1 & \ell_2 \\ 0 & 0 & 0 \end{pmatrix}^2, \quad (16)$$

for $\ell = 1, 2, \dots$. This result will have a great relevance for the practical implementation of the findings in the next sections.

4.2.1 Higher-order transforms

The general case of non linear transforms with $q \geq 3$ can be dealt with analogous lines, the main difference being the appearance of multiple integrals of spherical harmonics of order greater than 3, and hence so-called higher order Gaunt integrals and convolutions of Clebsch-Gordan coefficients. For brevity's sake, we provide only the basic details; we refer to [29] for a more detailed discussion on nonlinear transforms of Gaussian spherical harmonics. Here, we simply recall the definition of the multiple Gaunt integral (see [29], Remark 6.30 and Theorem 6.31), which is given by

$$\mathcal{G}(\ell_1, m_1; \dots, \ell_q, m_q; \ell, m) := \int_{S^2} Y_{\ell_1 m_1}(x) \dots Y_{\ell_q m_q}(x) Y_{\ell m}(x) d\sigma(x),$$

where the coefficients $\mathcal{G}(\ell_1, m_1; \dots, \ell_q, m_q; \ell, m)$ can be expressed as multiple convolution of Wigner/Clebsch-Gordan terms (see 15),

$$\begin{aligned} \mathcal{G}(\ell_1, m_1; \dots, \ell_q, m_q; \ell, m) &= (-1)^m \sqrt{\frac{(2\ell_1+1) \dots (2\ell_q+1)}{(4\pi)^{q-1} (2\ell+1)}} \\ &\times \sum_{\lambda_1 \dots \lambda_{q-2}} C_{\ell_1 0 \ell_2 0}^{\lambda_1 0} C_{\lambda_1 0 \ell_3 0}^{\lambda_2 0} \dots C_{\lambda_{q-2} 0 \ell_q 0}^{\ell 0} \sum_{\mu_1 \dots \mu_{q-2}} C_{\ell_1 m_1 \ell_2 m_2}^{\lambda_1 \mu_1} C_{\lambda_1 \mu_1 \ell_3 m_3}^{\lambda_2 \mu_2} \dots C_{\lambda_{q-2} \mu_{q-2} \ell_q m_q}^{\ell m}. \end{aligned}$$

Following also [29], eq. (6.40), let us introduce the shorthand notation

$$C_{\ell_1 0 \ell_2 0 \dots \ell_q 0}^{\lambda_1 \dots \lambda_{q-2} \ell 0} := C_{\ell_1 0 \ell_2 0}^{\lambda_1 0} C_{\lambda_1 0 \ell_3 0}^{\lambda_2 0} \dots C_{\lambda_{q-2} 0 \ell_q 0}^{\ell 0}, \quad \mathcal{C}(\ell_1, \dots, \ell_q, \ell) := \sum_{\lambda_1 \dots \lambda_{q-2}} \left\{ C_{\ell_1 0 \ell_2 0 \dots \ell_q 0}^{\lambda_1 \dots \lambda_{q-2} \ell 0} \right\}^2. \quad (17)$$

It should be noted that, from the unitary properties of Clebsch-Gordan coefficients

$$\sum_{\ell} \mathcal{C}(\ell_1, \dots, \ell_q, \ell) = \sum_{\lambda_1 \dots \lambda_{q-2}} \left\{ C_{\ell_1 0 \ell_2 0}^{\lambda_1 0} \right\}^2 \dots \sum_{\ell} \left\{ C_{\lambda_{q-2} 0 \ell_q 0}^{\ell 0} \right\}^2 = \dots = 1.$$

Lemma 10 *For general $q \geq 3$, the field $g_{j;q}(x)$ is zero-mean, finite variance and isotropic, with covariance function*

$$\begin{aligned} & \mathbb{E} \{g_{j;q}(x_1)g_{j;q}(x_2)\} \\ &= q! \sum_{\ell} \kappa^2(\ell) \sum_{\ell_1 \dots \ell_q} \mathcal{C}(\ell_1, \dots, \ell_q, \ell) \left[\prod_{k=1}^q b^2\left(\frac{\ell_k}{B^j}\right) \frac{2\ell_k + 1}{4\pi} C_{\ell_k} \right] P_{\ell}(\langle x_1, x_2 \rangle). \end{aligned}$$

Proof. We have

$$\begin{aligned} \mathbb{E} g_{j;q}^2(x) &= \mathbb{E} \left\{ \int_{S^2} \int_{S^2} K(\langle x, y_1 \rangle) K(\langle x, y_2 \rangle) H_q(\beta_j(y_1)) H_q(\beta_j(y_2)) dy_1 dy_2 \right\} \\ &= q! \int_{S^2} \int_{S^2} K(\langle x, y_1 \rangle) K(\langle x, y_2 \rangle) \left\{ \sum_{\ell} b^2\left(\frac{\ell}{B^j}\right) \frac{2\ell + 1}{4\pi} P_{\ell}(\langle y_1, y_2 \rangle) \right\}^q dy_1 dy_2. \end{aligned}$$

It is convenient here to view $T_{\ell}(x), \beta_j(x)$ as isonormal processes of the form

$$\begin{aligned} T_{\ell}(x) &= \int_{S^2} \sqrt{\frac{2\ell + 1}{4\pi}} C_{\ell} P_{\ell}(\langle x, y \rangle) dW(y), \\ \beta_j(x) &= \int_{S^2} \sum_{\ell} b\left(\frac{\ell}{B^j}\right) \sqrt{\frac{2\ell + 1}{4\pi}} C_{\ell} P_{\ell}(\langle x, y \rangle) dW(y), \end{aligned}$$

where $dW(y)$ denotes a Gaussian white noise measure on the sphere, whence

$$\begin{aligned} & H_q(\beta_j(x)) \\ &= \sum_{\ell_1 \dots \ell_q} b\left(\frac{\ell_1}{B^j}\right) \dots b\left(\frac{\ell_q}{B^j}\right) \sqrt{\prod_{i=1}^q \left\{ \frac{2\ell_i + 1}{4\pi} C_{\ell_i} \right\}} \\ & \quad \times \int_{S^2 \times \dots \times S^2} P_{\ell_1}(\langle x, y_1 \rangle) \dots P_{\ell_q}(\langle x, y_q \rangle) dW(y_1) \dots dW(y_q) \end{aligned}$$

and

$$\begin{aligned} g_{j;q}(z) &= \int_{S^2} \sum_{\ell} \kappa(\ell) \frac{2\ell + 1}{4\pi} P_{\ell}(\langle z, x \rangle) \sum_{\ell_1 \dots \ell_q} b\left(\frac{\ell_1}{B^j}\right) \dots b\left(\frac{\ell_q}{B^j}\right) \sqrt{\prod_{i=1}^q \left\{ \frac{2\ell_i + 1}{4\pi} C_{\ell_i} \right\}} \\ & \quad \times \int_{S^2 \times \dots \times S^2} P_{\ell_1}(\langle x, y_1 \rangle) \dots P_{\ell_q}(\langle x, y_q \rangle) dW(y_1) \dots dW(y_q) dx. \end{aligned}$$

It follows easily that

$$\begin{aligned} & \mathbb{E} \{g_{j;q}(z_1)g_{j;q}(z_2)\} = \\ &= \int_{S^2 \times S^2} \sum_{\ell_1 \ell_2} \frac{2\ell_1 + 1}{4\pi} \kappa(\ell_1) \frac{2\ell_2 + 1}{4\pi} \kappa(\ell_2) P_{\ell_1}(\langle z_1, x_1 \rangle) P_{\ell_2}(\langle z_2, x_2 \rangle) \\ & \quad \times \sum_{\ell_1 \dots \ell_q} b^2\left(\frac{\ell_1}{B^j}\right) \dots b^2\left(\frac{\ell_q}{B^j}\right) \sqrt{\prod_{i=1}^q \left\{ \frac{2\ell_i + 1}{4\pi} C_{\ell_i} \right\}} P_{\ell_1}(\langle x_1, x_2 \rangle) \dots P_{\ell_q}(\langle x_1, x_2 \rangle) dx_1 dx_2. \end{aligned}$$

Now write

$$\begin{aligned} & \frac{(2\ell_1 + 1) \dots (2\ell_q + 1)}{(4\pi)^q} P_{\ell_1}(\langle x_1, x_2 \rangle) \dots P_{\ell_q}(\langle x_1, x_2 \rangle) \\ &= \sum_{m_1 \dots m_q} Y_{\ell_1 m_1}(x_1) \dots Y_{\ell_q m_q}(x_1) \bar{Y}_{\ell_1 m_1}(x_2) \dots \bar{Y}_{\ell_q m_q}(x_2) \end{aligned}$$

so that

$$\begin{aligned} & \frac{(2\ell_1 + 1) \dots (2\ell_q + 1)}{(4\pi)^q} \int_{S^2 \times S^2} P_{\ell_1}(\langle z_1, x_1 \rangle) P_{\ell_2}(\langle z_2, x_2 \rangle) P_{\ell_1}(\langle x_1, x_2 \rangle) \dots P_{\ell_q}(\langle x_1, x_2 \rangle) dx_1 dx_2 \\ &= \sum_{\mu_1 \mu_2} \sum_{m_1 \dots m_q} \mathcal{G}(\ell_1, m_1; \dots \ell_q, m_q; \ell_1, \mu_1) \mathcal{G}(\ell_1, m_1; \dots \ell_q, m_q; \ell_2, \mu_2) \left\{ \frac{4\pi}{2\ell + 1} Y_{\ell_1 \mu_1}(z_1) \bar{Y}_{\ell_2 \mu_2}(z_2) \right\} \\ &= \frac{4\pi}{2\ell + 1} \sum_{\mu_1 \mu_2} Y_{\ell_1 \mu_1}(z_1) \bar{Y}_{\ell_2 \mu_2}(z_2) \delta_{\ell_1}^{\ell_2} \delta_{\mu_1}^{\mu_2} = P_{\ell_1}(\langle z_1, z_2 \rangle) . \end{aligned}$$

The general case $q \geq 3$ hence yields (see also [29], Theorem 7.5 for a related computation)

$$\begin{aligned} & \mathbb{E} g_{j;q}^2(x) = \\ & q! \sum_{\ell} \kappa^2(\ell) \sum_{\ell_1 \dots \ell_q} \mathcal{C}(\ell_1, \dots, \ell_q, \ell) b^2\left(\frac{\ell_1}{B^j}\right) \dots b^2\left(\frac{\ell_q}{B^j}\right) \frac{2\ell_1 + 1}{4\pi} \dots \frac{2\ell_q + 1}{4\pi} C_{\ell_1} \dots C_{\ell_q} , \end{aligned}$$

and

$$\begin{aligned} & \mathbb{E} \{g_{j;q}(x) g_{j;q}(y)\} \\ &= q! \sum_{\ell} \kappa^2(\ell) \sum_{\ell_1 \dots \ell_q} \mathcal{C}(\ell_1, \dots, \ell_q, \ell) b^2\left(\frac{\ell_1}{B^j}\right) \dots b^2\left(\frac{\ell_q}{B^j}\right) \frac{2\ell_1 + 1}{4\pi} \dots \frac{2\ell_q + 1}{4\pi} C_{\ell_1} \dots C_{\ell_q} P_{\ell}(\langle x_1, x_2 \rangle) , \end{aligned}$$

as claimed. ■

Remark 11 *It is immediately checked that the angular power spectrum of $g_{j;q}(y)$ is given by (see (17))*

$$C_{\ell;j,q} := q! \frac{4\pi}{2\ell + 1} \kappa^2(\ell) \sum_{\ell_1 \dots \ell_q} \mathcal{C}(\ell_1, \dots, \ell_q, \ell) \prod_{k=1}^q \left[b^2\left(\frac{\ell_k}{B^j}\right) \frac{2\ell_k + 1}{4\pi} C_{\ell_k} \right] . \quad (18)$$

As a special case, for $q = 2$ we recover the previous result (16)

$$\begin{aligned} C_{\ell;j,2} &= 2\kappa^2(\ell) \sum_{\ell_1 \ell_2} b^2\left(\frac{\ell_1}{B^j}\right) b^2\left(\frac{\ell_2}{B^j}\right) \frac{(2\ell_1 + 1)(2\ell_2 + 1)}{4\pi} C_{\ell_1} C_{\ell_2} \begin{pmatrix} \ell & \ell_1 & \ell_2 \\ 0 & 0 & 0 \end{pmatrix}^2 \\ &= 2\kappa^2(\ell) \frac{4\pi}{2\ell + 1} \sum_{\ell_1 \ell_2} \mathcal{C}(\ell_1, \ell_2, \ell) b^2\left(\frac{\ell_1}{B^j}\right) b^2\left(\frac{\ell_2}{B^j}\right) \frac{(2\ell_1 + 1)}{4\pi} \frac{(2\ell_2 + 1)}{4\pi} C_{\ell_1} C_{\ell_2} , \end{aligned} \quad (19)$$

because

$$\mathcal{C}(\ell_1, \ell_2, \ell) = \{C_{\ell_1 0 \ell_2 0}^{\ell 0}\}^2 = (2\ell + 1) \begin{pmatrix} \ell & \ell_1 & \ell_2 \\ 0 & 0 & 0 \end{pmatrix}^2 .$$

5 Weak Convergence

In this Section, we provide our main convergence results. It must be stressed that the convergence we study here is in some sense different from the standard theory as presented, for instance, by [11], but refers instead to the broader notion developed by [15], [14], see also [17], chapter 11.

We start first from the following Conditions:

Condition 12 *The angular power spectrum has the form*

$$C_\ell = G(\ell)\ell^{-\alpha}, \quad \ell = 1, 2, \dots,$$

where $\alpha > 2$ and $G(\cdot)$ is such that

$$\begin{aligned} 0 &< c_0 \leq G(\cdot) \leq d_0, \\ \left| \frac{d^r}{dx^r} G(x) \right| &\leq c_r \ell^{-r}, \quad r = 1, 2, \dots, M \in \mathbb{N}. \end{aligned}$$

Condition 13 *The Kernel $K(\cdot)$ and the field $\{\beta_j(\cdot)\}$ are such that, for all $j = 1, 2, 3, \dots$*

$$\text{Var} \left\{ \int_{S^2} K(\langle x, y \rangle) H_q(\tilde{\beta}_j(y)) dy \right\} = \sigma_j^2 B^{-2j}, \quad \text{for all } j = 1, 2, \dots$$

and there exist positive constants c_1, c_2 such that $c_1 \leq \sigma_j^2 \leq c_2$.

These assumptions are mild and it is easy to find many examples such that they are fulfilled. Heuristically, let us consider for instance $K(\langle x, y \rangle) \equiv 1$, so that we are really focussing on weighted averages of the field $\{H_q(\beta_j(y))\}$. It is readily checked that $\{H_q(\beta_j(y))\}$ is a polynomial of order $\simeq 2^{q(j+1)}$ and we can hence consider the following heuristic argument: we have

$$\begin{aligned} \int_{S^2} K(\langle x, y \rangle) H_q(\tilde{\beta}_j(y)) dy &= \int_{S^2} H_q(\tilde{\beta}_j(y)) dy \\ &= \sum_{k \in \mathcal{X}_j} H_q(\tilde{\beta}_j(\xi_{jk})) \lambda_{jk}, \end{aligned}$$

where $\{\xi_{jk}, \lambda_{jk}\}$ are a set of cubature points and weights (see [38], [9]); indeed, because the $\beta_j(\cdot)$ are band-limited (polynomial) functions, this Riemann sum approximations can be constructed to be exact, with weights λ_{jk} of order $\simeq B^{-2j}$. It is now known that under Condition 12, it is possible to establish a fundamental uncorrelation inequality which will play a crucial role in our proof below, see also [8], [24]. [36]; indeed, we have that for any $M \in \mathbb{N}$, there exist a constant C_M such that

$$\text{Cov} \left\{ H_q(\tilde{\beta}_j(\xi_{jk_1})), H_q(\tilde{\beta}_j(\xi_{jk_2})) \right\} \leq \frac{C_M q!}{\{1 + B^j d(\xi_{jk_1}, \xi_{jk_2})\}^{qM}},$$

entailing that the terms $H_q(\beta_j(\xi_{jk}))$ can be treated as asymptotically uncorrelated, for large j . Hence

$$\begin{aligned} \text{Var} \left\{ \sum_{k \in \mathcal{X}_j} H_q(\tilde{\beta}_j(\xi_{jk})) \lambda_{jk} \right\} &\simeq \sum_{k \in \mathcal{X}_j} \text{Var} \left\{ H_q(\tilde{\beta}_j(\xi_{jk})) \right\} \lambda_{jk}^2 \\ &\simeq C_q \sum_{k \in \mathcal{X}_j} \lambda_{jk}^2 \simeq C_q B^{-2j}, \end{aligned}$$

because $\sum_{k \in \mathcal{X}_j} \lambda_{jk} \simeq 4\pi$. For instance, for $q = 2$ we obtain

$$\begin{aligned} \text{Var} \left\{ \int_{S^2} (\tilde{\beta}_j^2(y) - 1) dy \right\} &= \text{Var} \left\{ \int_{S^2} \tilde{\beta}_j^2(y) dy \right\} \\ &= \frac{\text{Var} \left\{ \sum_{\ell} b^2(\frac{\ell}{B^j})(2\ell + 1) \hat{C}_{\ell} \right\}}{\left\{ \sum_{\ell} b^2(\frac{\ell}{B^j})(2\ell + 1) C_{\ell} \right\}^2} = \frac{\sum_{\ell} b^4(\frac{\ell}{B^j})(2\ell + 1)^2 \text{Var}(\hat{C}_{\ell})}{\left\{ \sum_{\ell} b^2(\frac{\ell}{B^j})(2\ell + 1) C_{\ell} \right\}^2} \\ &= \frac{2 \sum_{\ell=B^{j-1}}^{B^{j+1}} b^4(\frac{\ell}{B^j})(2\ell + 1) C_{\ell}^2}{\left\{ \sum_{\ell} b^2(\frac{\ell}{B^j})(2\ell + 1) C_{\ell} \right\}^2} \simeq \frac{B^{j(2-2\alpha)}}{\{B^{j(2-\alpha)}\}^2} \simeq B^{-2j}, \end{aligned}$$

as claimed.

5.1 Finite-dimensional distributions

The general technique we shall exploit to establish the Central Limit Theorem is based upon sharp bounds on normalized fourth-order cumulants. Note that, in view of results from [39], this will actually entail a stronger form of convergence, more precisely in total variation norm (see [39]).

We start by recalling that the field $\{\beta_j(\cdot)\}$ can be expressed in terms of the isonormal Gaussian process, e.g. as a stochastic integral

$$\beta_j(y) := \sum_{\ell} b(\frac{\ell}{B^j}) T_{\ell}(y) = \sum_{\ell} b(\frac{\ell}{B^j}) \sqrt{\frac{(2\ell + 1)C_{\ell}}{4\pi}} \int_{S^2} P_{\ell}(\langle y, z \rangle) W(dz),$$

where $W(A)$ is a white noise Gaussian measure on the sphere, which satisfies

$$\mathbb{E}W(A) = 0, \quad \mathbb{E}\{W(A)W(B)\} = \int_{A \cap B} dz, \quad \text{for all } A, B \in \mathcal{B}(S^2).$$

It thus follows immediately that the transformed process $\{H_q(\beta_j(\cdot))\}$ belongs to the q -th order Wiener chaos, see [39], [40] for more discussion and detailed definitions. Let us now recall the definition of the *total variation* distance between the laws of two random variables X and Z , which is given by

$$d_{TV}(X, Z) = \sup_{A \in \mathcal{B}(\mathbb{R})} |\Pr(W \in A) - \Pr(X \in A)|.$$

When Z is a standard Gaussian and X is a zero-mean, unit variance random variable which belongs to the q -th order Wiener chaos of a Gaussian measure, the following remarkable inequality holds for the total variation distance

$$d_{TV}(X, Z) \leq \sqrt{\frac{q-1}{3q} \text{cum}_4(X)} ,$$

see again [39], [40] for more discussion and a full proof.

Let us now introduce an isotropic zero-mean Gaussian process $f_{j;q}$, with the same covariance function as that of $g_{j;q}$. Our next result will establish the asymptotic convergence of the finite-dimensional distributions for $g_{j;q}$ and $f_{j;q}$. In particular, we have:

Lemma 14 *For any fixed vector (x_1, \dots, x_p) in S^2 , we have that*

$$d_{TV} \left(\left(\frac{g_{j;q}(x_1)}{\sqrt{\text{Var}(g_{j;q})}}, \dots, \frac{g_{j;q}(x_p)}{\sqrt{\text{Var}(g_{j;q})}} \right), \left(\frac{f_{j;q}(x_1)}{\sqrt{\text{Var}(g_{j;q})}}, \dots, \frac{f_{j;q}(x_p)}{\sqrt{\text{Var}(g_{j;q})}} \right) \right) = o(1),$$

as $j \rightarrow \infty$.

Proof. For notational simplicity, we shall focus on the univariate case; also, without loss of generality we normalize $\beta_j(\cdot)$ to have unit variance. In this case, the Nourdin-Peccati inequality ([39], [40]) can be restated as:

$$d_{TV} \left(\frac{g_{j;q}}{\sqrt{\text{Var}(g_{j;q})}}, N(0, 1) \right) \leq \sqrt{\frac{q-1}{3q} \text{cum}_4 \left(\frac{g_{j;q}}{\sqrt{\text{Var}(g_{j;q})}} \right)} . \quad (20)$$

In view of 20, for the Central Limit Theorem to hold we shall only need to study the limiting behaviour of the normalized fourth-order cumulant of $g_{j;q}$. Let us then consider

$$\begin{aligned} & \text{cum}_4 \{g_j(x)\} = \\ & \int_{\{S^2\}^{\otimes 4}} K(\langle x, y_1 \rangle) \dots K(\langle x, y_4 \rangle) \text{cum}_4 \{H_q(\beta_j(y_1)), \dots, H_q(\beta_j(y_4))\} dy_1 \dots dy_4 \\ & \leq C_1 \sup_{r=1, \dots, q-1} \int_{\{S^2\}^{\otimes 4}} |K(\langle x, y_1 \rangle) \dots K(\langle x, y_4 \rangle)| |\rho(y_1, y_2)|^{q-r} \\ & \quad \times |\rho(y_2, y_3)|^r |\rho(y_3, y_4)|^{q-r} |\rho(y_4, y_1)|^r dy_1 \dots dy_4 \\ & \leq C_2 \sup_{r=1, \dots, q-1} \int_{\{S^2\}^{\otimes 4}} |\rho(y_1, y_2)|^{q-r} |\rho(y_2, y_3)|^r |\rho(y_3, y_4)|^{q-r} |\rho(y_4, y_1)|^r dy_1 \dots dy_4 , \end{aligned}$$

where

$$\rho(y_1, y_2) = \frac{\sum_{\ell} b^2 \left(\frac{\ell}{B^j} \right) \frac{2\ell+1}{4\pi} C_{\ell} P_{\ell}(y_1, y_2)}{\sum_{\ell} b^2 \left(\frac{\ell}{B^j} \right) \frac{2\ell+1}{4\pi} C_{\ell}} \leq \frac{C_M}{\{1 + B^j d(y_1, y_2)\}^M} ,$$

in view of (12) and the uncorrelation inequality provided by [8], see also [24]. [36]. Now standard computations yield

$$\begin{aligned} \int_{S^2} |\rho(y_1, y_2)|^r dy_2 &\leq \int_{S^2} |\rho(y_1, y_2)| dy_2 \leq \int_{S^2} \frac{C_M}{\{1 + B^j d(y_1, y_2)\}^M} dy_2 \\ &\leq \int_{y_2: d(y_1, y_2) \leq 2B^{-j}} \frac{C_M}{\{1 + B^j d(y_1, y_2)\}^M} dy_2 + \int_{y_2: d(y_1, y_2) \geq 2B^{-j}} \frac{C_M}{\{1 + B^j d(y_1, y_2)\}^M} dy_2 \leq CB^{-2j} . \end{aligned}$$

Hence,

$$C_2 \sup_{r=1, \dots, q-1} \int_{\{S^2\}^{\otimes 4}} |\rho(y_1, y_2)|^{q-r} |\rho(y_2, y_3)|^r |\rho(y_3, y_4)|^{q-r} |\rho(y_4, y_1)|^r dy_1 \dots dy_4 \leq CB^{-6j}$$

and

$$\text{cum}_4 \left\{ \frac{g_{j;q}(x)}{\sqrt{\text{Var}(g_{j;q}(x))}} \right\} = O(B^{-2j}) ,$$

entailing that for every fixed $x \in S^2$,

$$d_{TV} \left(\frac{g_{j;q}}{\sqrt{\text{Var}(g_{j;q})}}, N(0, 1) \right) = O(B^{-2j}) ,$$

and hence the univariate Central Limit Theorem, as claimed. The proof in the multivariate case (see [41]) is entirely analogous and hence omitted for the sake of brevity. ■

5.2 Tightness

We now focus on asymptotic tightness for both sequences $\{g_{j;q}\}$ and $\{f_{j;q}\}$. We shall exploit the following criterion from [23]:

Proposition 15 ([23]) *Let $g_j : M \rightarrow D$ be a sequence of stochastic processes, where M is compact and D is complete and separable. Then $g_j \Rightarrow g$ if $g_j \rightarrow_{f.d.d.} g$ and (tightness condition)*

$$\lim_{h \rightarrow 0} \limsup_{j \rightarrow \infty} \mathbb{E} \left(\sup_{d(x,y) \leq h} |g_j(x) - g_j(y)| \wedge 1 \right) = 0 .$$

We are hence able to establish the following

Lemma 16 *For every $q \in \mathbb{N}$, the sequence $\{g_{j;q}\}$ and $\{f_{j;q}\}$ are tight.*

Proof. For any $x_1, x_2 \in S^2$, we have

$$\begin{aligned} |g_{j;q}(x_1) - g_{j;q}(x_2)| &\leq \int_{S^2} |K_j(\langle x_1, y \rangle) - K_j(\langle x_2, y \rangle)| |H_q(\beta_j(y))| dy \\ &\leq \int_{S^2} L_K d(x_1, x_2) |H_q(\beta_j(y))| dy , \text{ for all } j \geq 1 . \end{aligned}$$

where we used the Lipschitz bound

$$|K_j(\langle x_1, y \rangle) - K_j(\langle x_2, y \rangle)| \leq \sup_j L_K d(x_1, x_2) ,$$

which holds because $K_j(\cdot)$ is a finite-order polynomial. Then

$$\begin{aligned} \mathbb{E} \left(\sup_{d(x_1, x_2) \leq h} |g_{j;q}(x_1) - g_{j;q}(x_2)| \right) &\leq \sup_{d(x_1, x_2) \leq h} \int_{S^2} L_K d(x_1, x_2) \mathbb{E} |H_q(\beta_j(y))| dy \\ &\leq L_K h \int_{S^2} \sqrt{\mathbb{E} (H_q(\beta_j(y)))^2} dy \\ &= O(h) , \end{aligned}$$

whence the result for $g_{j;q}$ follows immediately. Likewise, for any $x_1, x_2 \in S^2$, we have

$$\begin{aligned} \mathbb{E} \left\{ \sup_{d(x_1, x_2) \leq \delta} |f_{j;q}(x_1) - f_{j;q}(x_2)| \right\} &= \mathbb{E} \left\{ \sup_{d(x_1, x_2) \leq \delta} \left| \sum_{\ell m} a_{\ell m}(f_{j;q}) \{Y_{\ell m}(x_1) - Y_{\ell m}(x_2)\} \right| \right\} \\ &\leq \sum_{\ell m} \{ \mathbb{E} |a_{\ell m}(f_{j;q})| \} \left\{ \sup_{d(x_1, x_2) \leq \delta} |\{Y_{\ell m}(x_1) - Y_{\ell m}(x_2)\}| \right\} . \end{aligned}$$

Now

$$\sup_{d(x_1, x_2) \leq \delta} |\{Y_{\ell m}(x_1) - Y_{\ell m}(x_2)\}| \leq c \ell^2 \delta ,$$

and

$$\sum_{\ell m} \{ \mathbb{E} |a_{\ell m}(f_{j;q})| \} \leq \sum_{\ell m} \sqrt{\mathbb{E} |a_{\ell m}(f_{j;q})|^2} = \sum_{\ell} (2\ell + 1) \sqrt{C_{\ell}(f_{j;q})}$$

and because $K(\cdot)$ is compactly supported in harmonic space (and hence, again, a finite-order polynomial)

$$\leq \left\{ \sum_{\ell}^{L_K} (2\ell + 1) \right\}^{1/2} \sqrt{\sum_{\ell}^{L_K} (2\ell + 1) C_{\ell}(f_{j;q})} \leq O(L_K) ,$$

whence

$$\mathbb{E} \left\{ \sup_{d(x_1, x_2) \leq \delta} |f_{j;q}(x_1) - f_{j;q}(x_2)| \right\} \leq C L_K^3 \delta ,$$

for some $C > 0$, uniformly over j , and thus the result follows. ■

5.3 Asymptotic Proximity of Distributions

Our discussion above shows that the finite-dimensional distributions of the non-Gaussian sequence of random fields $\{g_{j;q}\}$ converge to those of the Gaussian sequence $\{f_{j;q}\}$; moreover, both sequences are tight. However, the finite-dimensional distributions of neither processes converge to a well-defined limit.

In view of this situation, we need a broader notion of convergence than the one envisaged in standard treatment such as [11]; this extended form of convergence is provided by the notion of *Asymptotic Proximity*, or *Merging*, of distributions, as discussed for instance by [15], [14], [17], and others.

Definition 17 (Asymptotic Proximity of Distribution [15], [14], [17]) *Let g_n, f_n be two sequences of random elements in some metric space (X, ρ) , possibly defined on two different probability spaces. We say that the laws of g_n, f_n are asymptotically merging, or asymptotically proximal, (denoted as $g_n \Rightarrow f_n$) if and only if as $n \rightarrow \infty$*

$$|\mathbb{E}h(g_n) - \mathbb{E}h(f_n)| \rightarrow 0 ,$$

for all continuous and bounded functionals $h \in \mathcal{C}_b(X, \mathbb{R})$.

As discussed by [15], it is possible to provide a characterization of asymptotic proximity that extends in a natural way to standard weak convergence results.

Theorem 18 *Assume that $g_{j;q}, f_{j;q} \in \mathcal{C}_b(K, S)$, where K is compact and S is complete and separable. Then $g_{j;q}, f_{j;q}$ are asymptotically proximal if and only if they are both tight and their finite-dimensional distribution converge, i.e. for all $n \geq 1$, $x_1, \dots, x_n \in K$, we have that*

$$|\Pr \{(g_{j;q}(x_1), \dots, g_{j;q}(x_n)) \in A\} - \Pr \{(f_{j;q}(x_1), \dots, f_{j;q}(x_n)) \in A\}| \rightarrow 0 ,$$

for all $A \in \mathcal{B}(\mathbb{R}^n)$.

In view of the results provided in the previous subsection, we have hence established that

Theorem 19 *As $j \rightarrow \infty$*

$$g_{j;q} \Longrightarrow f_{j;q} ,$$

i.e. for all $h = h : \mathcal{C}(S^2, \mathbb{R}) \rightarrow \mathbb{R}$, h continuous and bounded, we have

$$|\mathbb{E}h(g_{j;q}) - \mathbb{E}h(f_{j;q})| \rightarrow 0 .$$

As a simple application of the asymptotic proximity result, we have

$$\mathbb{E} \left\{ \frac{\sup g_{j;q}}{1 + \sup g_{j;q}} \right\} \rightarrow \mathbb{E} \left\{ \frac{\sup f_{j;q}}{1 + \sup f_{j;q}} \right\} .$$

It should be noted that asymptotically proximal sequences do not enjoy all the same properties as in the standard weak convergence case. For instance, it is known that the Portmanteau Lemma does not hold in general, i.e. it is not true that, for every Borel set such that $\Pr \{g_j \in \partial A\} = \Pr \{f_j \in \partial A\} = 0$, we have

$$|\Pr \{g_j \in A\} - \Pr \{f_j \in A\}| \rightarrow 0 .$$

As a counterexample, it is enough to consider the sequences $f_j = -j^{-1}$ and $g_j = j^{-1}$. However, it is indeed possible to obtain more stringent characterizations when the subsequences are asymptotically Gaussian. We have the following

Proposition 20 *For every $A \in \mathcal{B}(\mathbb{R})$, we have that*

$$\left| \Pr \left\{ \sup_{x \in S^2} g_{j;q}(x) \in A \right\} - \Pr \left\{ \sup_{x \in S^2} f_{j;q} \in A \right\} \right| \rightarrow 0 .$$

Proof. We shall argue again by contradiction. Assume that there exists a subsequence j'_n such that for some $\varepsilon > 0$

$$\left| \Pr \left\{ \sup_{x \in S^2} g_{j'_n;q}(x) \in A \right\} - \Pr \left\{ \sup_{x \in S^2} f_{j'_n;q} \in A \right\} \right| > \varepsilon . \quad (21)$$

By relative compactness, there exists a subsequence j''_n and a limiting process $g_{\infty;q}$ such that

$$\left| \Pr \left\{ \sup_{x \in S^2} g_{j''_n;q}(x) \in A \right\} - \Pr \left\{ \sup_{x \in S^2} g_{\infty;q} \in A \right\} \right| \rightarrow 0 .$$

Likewise, consider $\{j'''_n\} \subset \{j''_n\}$; again by relative compactness there exist $f_{\infty;q}$ such that $f_{j'''_n;q} \Rightarrow f_{\infty;q}$ and hence

$$\left| \Pr \left\{ \sup_{x \in S^2} f_{j'''_n;q}(x) \in A \right\} - \Pr \left\{ \sup_{x \in S^2} f_{\infty;q} \in A \right\} \right| \rightarrow 0 .$$

Note that $f_{\infty;q}, g_{\infty;q}$ are isotropic and continuous Gaussian random fields, whence the supremum is necessarily a continuous random variable, whence no problems with non-zero boundary probabilities arise. Now the finite-dimensional distributions are a determining class, whence the two Gaussian processes $f_{\infty;q}, g_{\infty;q}$ must necessarily have the same distribution. Hence

$$\left| \Pr \left\{ \sup_{x \in S^2} f_{j'''_n;q}(x) \in A \right\} - \Pr \left\{ \sup_{x \in S^2} g_{j'''_n;q}(x) \in A \right\} \right| \rightarrow 0 ,$$

yielding a contradiction with 21. ■

This result immediately suggests two alternative ways to achieve the ultimate goal of this paper, e.g. the evaluation of excursion probabilities on the non-Gaussian sequence of random fields $\{g_{j;q}\}$. On one hand, it follows immediately that these probabilities may be evaluated by simulations, by simply sampling realizations of a Gaussian field with known angular power spectrum; for $q = 2$, for example, $f_{j;q}$ is simply a Gaussian process with angular power spectrum given by (19). There exist now very efficient techniques, based on packages such as HealPix ([20]), for the numerical simulation of Gaussian fields with a given power spectra; here the only burdensome step can be the numerical evaluation of expressions like (19), but this is in any case much faster and simpler than the Monte Carlo evaluation of smoothed non-Gaussian fields. Therefore our result has an immediate applied relevance.

One can try, however, to be more ambitious than this, and verify whether these excursion probabilities can indeed be evaluated analytically, rather than by Gaussian simulations. This is in fact the purpose of the next, and final, Section.

6 Asymptotics for the excursion probabilities

The purpose of this final Section is to show how the previous weak convergence results allow for very neat characterizations of excursion probabilities, even in non-Gaussian circumstances. In particular, if we consider the normalized version $\tilde{g}_{j;q}$ of the field $g_{j;q}$, such that $\tilde{g}_{j;q}(x) = (\text{Var}(g_{j;q}(x)))^{-1/2} g_{j;q}(x)$, then the following Theorem can be proved.

Theorem 21 *There exists a constant $\alpha > 1$, such that*

$$\limsup_{j \rightarrow \infty} \left| \Pr \left\{ \sup_{x \in S^2} \tilde{g}_{j;q}(x) > u \right\} - \{2(1 - \Phi(u)) + 2u\phi(u)\lambda_{j;q}\} \right| \leq \exp \left(-\frac{\alpha u^2}{2} \right),$$

where (see (18))

$$\lambda_{j;q} = \frac{\sum_{\ell=1}^L \frac{2\ell+1}{4\pi} C_{\ell;j,q} P'_\ell(1)}{\sum_{\ell=1}^L \frac{2\ell+1}{4\pi} C_{\ell;j,q}}. \quad (22)$$

In order to establish the above Theorem, we shall need to fine tune Theorem 14.3.3 of [1] to our needs. Let us begin with writing $f_{j;q}$ as a mean zero Gaussian random field on S^2 whose covariance function matches with that of $\tilde{g}_{j;q}$; note that in the previous sections, $f_{j;q}$ denoted the Gaussian random field whose mean and covariance matched with that of $g_{j;q}$, whereas in this section, the new $f_{j;q}$ is basically the normalized version of the old one so as to have unit variance. Then, for each $x_0 \in S^2$, define

$$\begin{aligned} \hat{f}_{j;q}^{x_0}(x) = & \frac{1}{1 - \rho(x, x_0)} \left\{ f_{j;q}(x) - \rho(x, x_0) f_{j;q}(x_0) \right. \\ & - \text{Cov}(f_{j;q}(x), \frac{\partial}{\partial \vartheta} f_{j;q}(x_0)) \text{Var}(\frac{\partial}{\partial \vartheta} f_{j;q}(x)) \frac{\partial}{\partial \vartheta} f_{j;q}(x) \\ & \left. - \text{Cov}(f_{j;q}(x), \frac{\partial}{\sin \vartheta \partial \phi} f_{j;q}(x_0)) \text{Var}(\frac{\partial}{\sin \vartheta \partial \phi} f_{j;q}(x)) \frac{\partial}{\sin \vartheta \partial \phi} f_{j;q}(x) \right\}, \end{aligned}$$

where $\rho(x, x_0) = E(f_{j;q}(x)f_{j;q}(x_0))$. Next define,

$$\mu_j^+ = \sup_{x_0} E \left(\sup_{x \neq x_0} \hat{f}_{j;q}^{x_0}(x) \right)$$

and,

$$\sigma_j^2 = \sup_{x_0} \sup_{x \neq x_0} \text{Var}(\hat{f}_{j;q}^{x_0}(x)).$$

Then, under suitable regularity conditions on the kernel K , we have the following Proposition, whose proof is deferred to the Appendix.

Proposition 22 *Under the assumption that the kernel K appearing in the definition of $\tilde{g}_{j;q}$ is of the form*

$$K(x, y) = \sum_{i=1}^{\ell} k_i \frac{2i+1}{4\pi} P_i(\langle x, y \rangle), \quad (23)$$

for some fixed positive integer ℓ , the field $\widehat{f}_{j;q}^{x_0}$ satisfies the following:

$$\mathbb{E}(\widehat{f}_{j;q}^{x_0}(x_2) - \widehat{f}_{j;q}^{x_0}(x_1))^2 \leq K(\ell)|x_2 - x_1|, \quad (24)$$

where $K(\ell)$ depends on q and ℓ , but does not depend on j .

We are now in a position to provide the following

Proposition 23 *For large enough u , there exists $\alpha > 1$ such that, uniformly over j*

$$\left| \Pr \left\{ \sup_{x \in S^2} f_{j;q}(x) > u \right\} - \mathbb{E} \mathcal{L}_0(A_u(f_{j;q}, S^2)) \right| \leq \exp\left(-\frac{\alpha u^2}{2}\right),$$

where $\mathbb{E} \mathcal{L}_0(A_u(f_{j;q}, S^2)) = 2(1 - \Phi(u)) + 2u\phi(u)\lambda_{j;q}$.

Proof. From p. 371 of [1], we know that for $u \geq \mu_j^+$

$$\begin{aligned} & \left| \Pr \left\{ \sup_{x \in S^2} f_{j;q}(x) > u \right\} - \mathbb{E} \mathcal{L}_0(A_u(f_{j;q}, S^2)) \right| \\ & \leq K u e^{-\frac{(u - \mu_j^+)^2}{2} \left(1 + \frac{1}{2\sigma_j^2}\right)} \sum_{i=0}^2 \left\{ \mathbb{E} \left| \det_i(-\nabla^2 f_{j;q} - f_{j;q} I_2) \right|^2 \right\}^{1/2}, \end{aligned} \quad (25)$$

where I_2 is the 2×2 identity matrix, K is a constant not depending on j , and \det_i of a matrix is the sum over all the i -minors of the matrix under consideration.

Our goal is to get a uniform bound for the right hand side of (25). Clearly, $\sum_{i=0}^2 \mathbb{E} \left| \det_i(-\nabla^2 f_{j;q} - f_{j;q} I_2) \right|^2$ is bounded above by a universal constant. To get a uniform bound for μ_j^+ , we shall resort to the standard techniques on estimating suprema of a Gaussian random field using the method of metric entropy. *En passant*, a uniform bound on σ_j^2 is also obtained.

Applying Theorem 1.4.1 of [1] to the random fields $f_{j;q}$, and assuming that the equation (24) is satisfied, we conclude that the metric entropy of $f_{j;q}$ can be uniformly bounded above, which in turn implies that $\sup_j \mu_j^+ < \infty$.

Similarly, a uniform lower bound on σ_j^2 is also obtained; for the sake of brevity, we shall present its proof in the Appendix. Thus for $u \geq \sup_j \mu_j^+$,

$$\left| \Pr \left\{ \sup_{x \in S^2} f_{j;q}(x) > u \right\} - \mathbb{E} \mathcal{L}_0(A_u(f_{j;q}, S^2)) \right| \leq K u e^{-\frac{\alpha(u - \mu^+)^2}{2}},$$

where the K appearing here is different from the earlier one. Finally, note that the linear term u can also be ignored by choosing a smaller α in the exponent, which completes the proof of the theorem. ■

Proof of Theorem 21. From the results of previous section, we know that for any fixed u

$$\lim_{j \rightarrow \infty} \left| \Pr \left\{ \sup_{x \in S^2} \tilde{g}_{j;q}(x) > u \right\} - \Pr \left\{ \sup_{x \in S^2} f_{j;q}(x) > u \right\} \right| = 0.$$

Combining this with Proposition 23, we get the desired result. ■

References

- [1] **Adler, R. J. and Taylor, J. E., (2007)** *Random Fields and Geometry*, Springer.
- [2] **Adler, R. J. and Taylor, J. E. (2011)** *Topological Complexity of Smooth Random Functions*. Lecture Notes in Mathematics, 2019, Springer, Heidelberg.
- [3] **Adler, R. J., Samorodnitsky, G. and Taylor, J. E. (2010)** Excursion Sets of Three Classes of Stable Random Fields. *Adv. in Appl. Probab.* 42, no. 2, 293–318.
- [4] **Adler, R., Moldavskaya, E. and Samorodnitsky, G. (2013)** On the Existence of Paths Between Points in High Level Excursion Sets of Gaussian Random Fields, *Annals of Probability*, in press.
- [5] **Anderes, E., (2010)** On the Consistent Separation of Scale and Variance for Gaussian Random Fields, *Annals of Statistics*, 38, no. 2, 870–893
- [6] **Azaïs, J.-M., Wschebor, M. (2005)** On the Distribution of the Maximum of a Gaussian Field with d Parameters. *Ann. Appl. Probab.* 15 no. 1A, 254–278.
- [7] **Azaïs, J.-M., Wschebor, M. (2009)** Level Sets and Extrema of Random Processes and Fields. John Wiley & Sons, Inc., Hoboken, NJ.
- [8] **Baldi, P., Kerkycharian, G., Marinucci, D. and Picard, D. (2009)** Asymptotics for Spherical Needlets, *Annals of Statistics*, Vol. 37, No. 3, 1150–1171
- [9] **Baldi, P., Kerkycharian, G., Marinucci, D. and Picard, D. (2009)** Subsampling Needlet Coefficients on the Sphere, *Bernoulli*, Vol. 15, 438–463, arXiv: 0706.4169
- [10] **Bennett, C.L. et al. (2012)** Nine-Year WMAP Observations: Final Maps and Results, arXiv:1212.5225
- [11] **Billingsley, P. (1968)** *Convergence of Probability Measures*. John Wiley & Sons, Inc., New York-London-Sydney
- [12] **Blum, G., Gnutzmann, S. and Smilansky, U. (2002)** Nodal Domains Statistics: A Criterion for Quantum Chaos. *Physical Review Letters*, 88, 114101.
- [13] **Cheng, D. and Xiao, Y. (2012)** The Mean Euler Characteristic and Excursion Probability of Gaussian Random Fields with Stationary Increments, arXiv:1211.6693
- [14] **D’Aristotile, A., Diaconis, P., Freedman, D. (1988)** On Merging of Probabilities. *Sankhyā* Ser. A 50 no. 3, 363–380.

- [15] **Davydov, Y. and Rotar, V. (2009)** On Asymptotic Proximity of Distributions, *J. Theoret. Probab.* 22 (2009), no. 1, 82–98
- [16] **Dodelson, S. (2003)** *Modern Cosmology*, Academic Press
- [17] **Dudley, R. M. (2002)** *Real Analysis and Probability*. Revised reprint of the 1989 original. Cambridge Studies in Advanced Mathematics, 74. Cambridge University Press, Cambridge.
- [18] **Durrer, R. (2008)** *The Cosmic Microwave Background*, Cambridge University Press.
- [19] **Geller, D. and Mayeli, A. (2009)** Continuous Wavelets on Manifolds, *Math. Z.*, Vol. 262, pp. 895-927, arXiv: math/0602201
- [20] **Gorski, K.M., Hivon, E., Banday, A.J., Wandelt, B.D., Hansen, F.K., Reinecke, M, Bartelman, M. (2005)**, HEALPix – a Framework for High Resolution Discretization, and Fast Analysis of Data Distributed on the Sphere, *Astrophys. J.* 622:759-771
- [21] **Hansen, F. K., Banday, A. J., Górski, K. M., Eriksen, H. K., and Lilje, P. B. (2009)** Power Asymmetry in Cosmic Microwave Background Fluctuations from Full Sky to Sub-Degree Scales: Is the Universe Isotropic?, *The Astrophysical Journal*, Volume 704, Issue 2, pp. 1448-1458
- [22] **Hotelling, H. (1939)** Tubes and Spheres in n -Spaces and a Class of Statistical Problems, *Amer. J. Math.*, 61: 440-460.
- [23] **Kallenberg, O. (1997)** *Foundations of Modern Probability*. Probability and its Applications (New York). Springer-Verlag, New York
- [24] **Lan, X. and Marinucci, D. (2008)** On the Dependence Structure of Wavelet Coefficients for Spherical Random Fields, *Stochastic Processes and their Applications*, 119, 3749-3766, arXiv:0805.4154
- [25] **Larson, D. et al. (2011)** Seven-Year Wilkinson Microwave Anisotropy Probe (WMAP) Observations: Power Spectra and WMAP-Derived Parameters, *Astrophysical Journal Supplement Series*, 192:16.
- [26] **Leonenko, N. (1999)** *Limit Theorems for Random Fields with Singular Spectrum*, Mathematics and its Applications, 465. Kluwer Academic Publishers, Dordrecht
- [27] **Loh, W.-L. (2005)** Fixed-Domain Asymptotics for a Subclass of Matérn-type Gaussian Random Fields, *Annals of Statistics*, Vol. 33, No. 5, 2344-2394
- [28] **Malyarenko, A. (2012)**, *Invariant Random Fields on Spaces with a Group Action*, Probability and its Applications, Springer.

- [29] **Marinucci, D. and Peccati, G. (2011)** *Random Fields on the Sphere. Representation, Limit Theorem and Cosmological Applications*, Cambridge University Press
- [30] **Marinucci, D. and Peccati, G. (2012)** Mean Square Continuity on Homogeneous Spaces of Compact Groups, arXiv:1210.7676.
- [31] **Marinucci, D., Pietrobon, D., Balbi, A., Baldi, P., Cabella, P., Kerkycharian, G., Natoli, P. Picard, D., Vittorio, N., (2008)** Spherical Needlets for CMB Data Analysis, *Monthly Notices of the Royal Astronomical Society*, Volume 383, Issue 2, pp. 539-545
- [32] **Marinucci, D. and Wigman, I. (2011)** On the Excursion Sets of Spherical Gaussian Eigenfunctions, *Journal of Mathematical Physics*, 52, 093301, arXiv 1009.4367
- [33] **Marinucci, D. and Wigman, I. (2011)** The Defect Variance of Random Spherical Harmonics, *Journal of Physics A: Mathematical and Theoretical*, 44, 355206, arXiv:1103.0232
- [34] **Marinucci, D. and Wigman, I. (2012)**, On Nonlinear Functionals of Random Spherical Eigenfunctions, preprint, arXiv: 1209.1841.
- [35] **Matsubara, T. (2010)** Analytic Minkowski Functionals of the Cosmic Microwave Background: Second-order Non-Gaussianity with Bispectrum and Trispectrum, *Physical Review D*, 81:083505
- [36] **Mayeli, A. (2010)**, Asymptotic Uncorrelation for Mexican Needlets, *J. Math. Anal. Appl.* Vol. 363, Issue 1, pp. 336-344, arXiv: 0806.3009
- [37] **McEwen, J. D., Vielva, P., Wiaux, Y., Barreiro, R. B., Cayón, I., Hobson, M. P., Lasenby, A. N., Martínez-González, E. and Sanz, J. L. (2007)** Cosmological Applications of a Wavelet Analysis on the Sphere. *J. Fourier Anal. Appl.* 13, no. 4, 495–510.
- [38] **Narcowich, F.J., Petrushev, P. and Ward, J.D. (2006a)** Localized Tight Frames on Spheres, *SIAM Journal of Mathematical Analysis* Vol. 38, pp. 574–594
- [39] **Nourdin, I. and Peccati, G. (2009)** Stein’s Method on Wiener Chaos, *Probability Theory and Related Fields*, 145, no. 1-2, 75–118.
- [40] **Nourdin, I. and Peccati, G. (2012)** *Normal Approximations Using Malliavin Calculus: from Stein’s Method to Universality*, Cambridge University Press
- [41] **Peccati, G., and Tudor, C. (2005)** Gaussian Limits for Vector-Valued Multiple Stochastic Integrals, *Séminaire de Probabilités XXXVIII, Lecture Notes in Mathematics*, 1857, pp. 247–262, Springer, Berlin.

- [42] **Pietrobon, D., Balbi, A., Marinucci, D. (2006)** Integrated Sachs-Wolfe Effect from the Cross Correlation of WMAP3 Year and the NRAO VLA Sky Survey Data: New Results and Constraints on Dark Energy, *Physical Review D*, id. D:74, 043524
- [43] **Pietrobon, D., Amblard, A., Balbi, A., Cabella, P., Cooray, A., Marinucci, D. (2008)** Needlet Detection of Features in WMAP CMB Sky and the Impact on Anisotropies and Hemispherical Asymmetries, *Physical Review D*, D78 103504
- [44] **Rudjord, O., Hansen, F.K., Lan, X., Liguori, M. Marinucci, D., Matarrese, S. (2010)** Directional Variations of the Non-Gaussianity Parameter f_{NL} , *Astrophysical Journal*, Volume 708, Issue 2, pp. 1321-1325, arXiv: 0906.3232
- [45] **Starck, J.-L., Murtagh, F. and Fadili, J. (2010)** *Sparse Image and Signal Processing: Wavelets, Curvelets, Morphological Diversity*, Cambridge University Press.
- [46] **Stein, M.L. (1999)** *Interpolation of Spatial Data. Some Theory for Kriging*. Springer Series in Statistics. Springer-Verlag, New York, 1999.
- [47] **Stein, E.M. and Weiss, G. (1971)** *Introduction to Fourier Analysis on Euclidean Spaces*. Princeton University Press
- [48] **Taylor, J.E. and Adler, R. J. (2003)** Euler characteristics for Gaussian fields on manifolds. *Ann. Probab.* 31, no. 2, 533–563.
- [49] **Taylor, J.E., Takemura, A., Adler, R. J. (2005)** Validity of the Expected Euler Characteristic Heuristic. *Ann. Probab.* 33, no. 4, 1362–1396.
- [50] **Taylor, J.E. and Adler, R.J. (2009)** Gaussian Processes, Kinematic Formulae and Poincaré’s Limit. *Ann. Probab.* 37, no. 4, 1459–1482.
- [51] **Taylor, J.E. and Vadhvani, S. (2012)** Random Fields and the Geometry of Wiener Space, *Ann. Probab.*, in press, arXiv: 1105.3839
- [52] **Varshalovich, D.A., A.N. Moskalev, and V.K. Khersonskii (1988)**, *Quantum Theory of Angular Momentum*, World Scientific
- [53] **Wang, D. and Loh, W.-L. (2011)** , On Fixed-Domain Asymptotics and Covariance Tapering in Gaussian Random Field Models, *Electronic Journal of Statistics*, Vol. 5, 238-269
- [54] **Weyl, H. (1939)**, On the Volume of Tubes, *Amer. J. Math.*, 61: 461-472
- [55] **Wigman, I. (2009)** On the Distribution of the Nodal Sets of Random Spherical Harmonics. *Journal of Mathematical Physics* 50, no. 1, 013521, 44 pp.

- [56] **Wigman, I. (2010)** Fluctuation of the Nodal Length of Random Spherical Harmonics, *Communications in Mathematical Physics*, Volume 298, n. 3, 787-831

7 Appendix

Here we present the proof of Proposition 22; we shall also obtain a uniform lower bound on σ_j^2 . For notational simplicity and without loss of generality, we take the coefficients $\{k_i \frac{2i+1}{4\pi}\}$ in (23) to be identically equal to one.

Writing $\rho(x, y) = \text{cov}(f_{j;q}(x), f_{j;q}(y))$, and $\partial_{\phi_x}, \partial_{\theta_x}$ as directional derivatives at x in the normalized spherical coordinate directions we have

$$\begin{aligned}
& \text{cov}\left(\widehat{f}_{j;q}^{x_0}(x_1), \widehat{f}_{j;q}^{x_0}(x_2)\right) \\
&= \frac{1}{(1 - \rho(x_0, x_1))(1 - \rho(x_0, x_2))} \left(\rho(x_1, x_2) - \rho(x_0, x_1)\rho(x_0, x_2) \right. \\
&\quad - \text{cov}(f_{j;q}(x_1), \partial_{\theta_{x_0}} f_{j;q}(x_0)) \text{cov}(f_{j;q}(x_2), \partial_{\theta_{x_1}} f_{j;q}(x_1)) \text{cov}(\partial_{\theta_{x_1}} f_{j;q}(x_1), \partial_{\theta_{x_1}} f_{j;q}(x_1)) \\
&\quad - \text{cov}(f_{j;q}(x_1), \partial_{\phi_{x_0}} f_{j;q}(x_0)) \text{cov}(f_{j;q}(x_2), \partial_{\phi_{x_1}} f_{j;q}(x_1)) \text{cov}(\partial_{\phi_{x_1}} f_{j;q}(x_1), \partial_{\phi_{x_1}} f_{j;q}(x_1)) \\
&\quad - \rho(x_0, x_1)\rho(x_0, x_2) + \rho(x_0, x_1)\rho(x_0, x_2)\rho(x_0, x_0) \\
&\quad + \rho(x_0, x_2) \text{cov}(f_{j;q}(x_1), \partial_{\theta_{x_0}} f_{j;q}(x_0)) \text{cov}(f_{j;q}(x_0), \partial_{\theta_{x_1}} f_{j;q}(x_1)) \text{cov}(\partial_{\theta_{x_1}} f_{j;q}(x_1), \partial_{\theta_{x_1}} f_{j;q}(x_1)) \\
&\quad + \rho(x_0, x_2) \text{cov}(f_{j;q}(x_1), \partial_{\phi_{x_0}} f_{j;q}(x_0)) \text{cov}(f_{j;q}(x_0), \partial_{\phi_{x_1}} f_{j;q}(x_1)) \text{cov}(\partial_{\phi_{x_1}} f_{j;q}(x_1), \partial_{\phi_{x_1}} f_{j;q}(x_1)) \\
&\quad - \text{cov}(f_{j;q}(x_2), \partial_{\theta_{x_0}} f_{j;q}(x_0)) \text{cov}(\partial_{\theta_{x_2}} f_{j;q}(x_2), f_{j;q}(x_1)) \text{cov}(\partial_{\theta_{x_2}} f_{j;q}(x_2), \partial_{\theta_{x_2}} f_{j;q}(x_2)) \\
&\quad + \rho(x_0, x_1) \text{cov}(f_{j;q}(x_2), \partial_{\theta_{x_0}} f_{j;q}(x_0)) \text{cov}(\partial_{\theta_{x_2}} f_{j;q}(x_2), f_{j;q}(x_0)) \text{cov}(\partial_{\theta_{x_2}} f_{j;q}(x_2), \partial_{\theta_{x_2}} f_{j;q}(x_2)) \\
&\quad + (\text{var}(\partial_{\theta_{x_1}} f_{j;q}(x_1)))^2 \text{cov}(f_{j;q}(x_1), \partial_{\theta_{x_0}} f_{j;q}(x_0)) \text{cov}(f_{j;q}(x_2), \partial_{\theta_{x_0}} f_{j;q}(x_0)) \\
&\quad \left. \text{cov}(\partial_{\theta_{x_1}} f_{j;q}(x_1), \partial_{\theta_{x_2}} f_{j;q}(x_2)) \right. \\
&\quad + \text{var}(\partial_{\theta_{x_1}} f_{j;q}(x_1)) \text{var}(\partial_{\phi_{x_2}} f_{j;q}(x_2)) \text{cov}(f_{j;q}(x_1), \partial_{\phi_{x_0}} f_{j;q}(x_0)) \\
&\quad \text{cov}(f_{j;q}(x_2), \partial_{\theta_{x_0}} f_{j;q}(x_0)) \text{cov}(\partial_{\theta_{x_1}} f_{j;q}(x_1), \partial_{\phi_{x_2}} f_{j;q}(x_2)) \\
&\quad - \text{cov}(f_{j;q}(x_2), \partial_{\phi_{x_0}} f_{j;q}(x_0)) \text{var}(\partial_{\phi_{x_2}} f_{j;q}(x_2)) \text{cov}(\partial_{\phi_{x_2}} f_{j;q}(x_2), f_{j;q}(x_1)) \\
&\quad + \rho(x_0, x_1) \text{cov}(f_{j;q}(x_2), \partial_{\phi_{x_0}} f_{j;q}(x_0)) \text{var}(\partial_{\phi_{x_2}} f_{j;q}(x_2)) \text{cov}(\partial_{\phi_{x_2}} f_{j;q}(x_2), f_{j;q}(x_0)) \\
&\quad + \text{var}(\partial_{\theta_{x_1}} f_{j;q}(x_1)) \text{var}(\partial_{\phi_{x_2}} f_{j;q}(x_2)) \text{cov}(f_{j;q}(x_1), \partial_{\theta_{x_0}} f_{j;q}(x_0)) \\
&\quad \text{cov}(f_{j;q}(x_2), \partial_{\phi_{x_0}} f_{j;q}(x_0)) \text{cov}(\partial_{\theta_{x_1}} f_{j;q}(x_1), \partial_{\phi_{x_2}} f_{j;q}(x_2)) \\
&\quad + \left(\text{var}(\partial_{\phi_{x_1}} f_{j;q}(x_1)) \right)^2 \text{cov}(f_{j;q}(x_1), \partial_{\phi_{x_0}} f_{j;q}(x_0)) \text{cov}(f_{j;q}(x_2), \partial_{\phi_{x_0}} f_{j;q}(x_0)) \\
&\quad \left. \text{cov}(\partial_{\phi_{x_1}} f_{j;q}(x_1), \partial_{\phi_{x_2}} f_{j;q}(x_2)) \right)
\end{aligned}$$

Note that $\rho(x_1, x_2)$ can be assumed to have $P_l(\langle x_1, x_2 \rangle)$ as the leading polynomial (uniform over all j). Then, putting $x_1 = x_2$ in the above computation,

and going through some more (but simple) calculations, one can show that there exists a constant $M > 0$ such that $\text{Var}(\hat{f}_{j;q}^{x_0}(x)) \geq M$ uniformly over all j , which in turn, together with the assumption of isotropy, proves that $\sigma_j^2 \geq M'$, for some $M' > 0$.

Next, to prove the Proposition 22 we begin with

$$\mathbb{E} \left(\hat{f}_{j;q}^{x_0}(x_2) - \hat{f}_{j;q}^{x_0}(x_1) \right)^2 = \text{var}(\hat{f}_{j;q}^{x_0}(x_1)) + \text{var}(\hat{f}_{j;q}^{x_0}(x_2)) - 2\text{cov} \left(\hat{f}_{j;q}^{x_0}(x_1), \hat{f}_{j;q}^{x_0}(x_2) \right).$$

We shall analyze each pair of the terms in the above expression separately. Let us, for instance consider (together) one of the, seemingly, more involved term of the expression which is the last term of the covariance and the corresponding term in $\text{var}(\hat{f}_{j;q}^{x_0}(x_1))$. At the expense of introducing more notation, let us write $C_{\ell;\phi\phi} = \text{var}(\partial_{\phi_x} f_{j;q}(x))$ (note that due to isotropy, the variance does not depend on the spatial point x), then the difference between the last term of $\text{Var}(\hat{f}_{j;q}^{x_0}(x_1))$ and the last term of $\text{Cov}(\hat{f}_{j;q}^{x_0}(x_1), \hat{f}_{j;q}^{x_0}(x_2))$, can be written as, for all $x_1, x_2 \in (B(x_0, \epsilon))^c$ i.e., outside a ball of size ϵ around the point x_0 , we shall have

$$\begin{aligned} & \frac{1}{(1 - \rho(x_0, x_1))^2 (1 - \rho(x_0, x_2))} \\ & \times \left(C_{\ell;\phi\phi}^3 \left(\text{cov}(f_{j;q}(x_1), \partial_{\phi_{x_0}} f_{j;q}(x_0)) \right)^2 (1 - \rho(x_0, x_2)) - C_{\ell;\phi\phi}^2 \text{cov}(f_{j;q}(x_1), \partial_{\phi_{x_0}} f_{j;q}(x_0)) \right. \\ & \left. \text{cov}(f_{j;q}(x_2), \partial_{\phi_{x_0}} f_{j;q}(x_0)) \text{cov}(\partial_{\phi_{x_1}} f_{j;q}(x_1), \partial_{\phi_{x_2}} f_{j;q}(x_2)) (1 - \rho(x_0, x_1)) \right) \\ = & \frac{C_{\ell;\phi\phi}^2 \partial_{\phi_{x_0}} \rho(\langle x_1, x_0 \rangle)}{(1 - \rho(x_0, x_1))^2 (1 - \rho(x_0, x_2))} \\ & \times \left(C_{\ell;\phi\phi} (1 - \rho(x_0, x_2)) \partial_{\phi_{x_0}} \rho(x_1, x_0) - (1 - \rho(x_0, x_1)) \partial_{\phi_{x_0}} \rho(x_2, x_0) \partial_{\phi_{x_1}} \partial_{\phi_{x_2}} \rho(x_1, x_2) \right) \\ = & \frac{C_{\ell;\phi\phi}^2 \partial_{\phi_{x_0}} \rho(\langle x_1, x_0 \rangle)}{(1 - \rho(x_0, x_1))^2 (1 - \rho(x_0, x_2))} \\ & \times \left(\left(\partial_{\phi_{x_0}} \rho(x_1, x_0) - \partial_{\phi_{x_0}} \rho(x_2, x_0) \right) C_{\ell;\phi\phi} (1 - \rho(x_0, x_2)) \right. \\ & \left. + \partial_{\phi_{x_0}} \rho(x_2, x_0) \left(C_{\ell;\phi\phi} (1 - \rho(x_0, x_2)) - (1 - \rho(x_0, x_1)) \partial_{\phi_{x_1}} \partial_{\phi_{x_2}} \rho(x_1, x_2) \right) \right) \end{aligned}$$

Recall that the covariance function ρ does depend on j , but since we are assuming the kernel $K(x, y)$ to have finite expansion, thus the corresponding Legendre polynomial expansion of $\rho(x_1, x_2)$ can be assumed to have a $P_\ell(\langle x_1, x_2 \rangle)$ (uniform over j) which is the leading polynomial. Then, taking the modulus of the above expression, and considering all $x_1, x_2 \in (B(x_0, \epsilon))^c$ i.e., outside a ball of size ϵ around the point x_0 , we shall have

$$\begin{aligned}
& \left| \frac{C_{\ell;\phi\phi}^2 \partial_{\phi_{x_0}} P_\ell(\langle x_1, x_0 \rangle)}{[1 - P_\ell(\langle x_0, x_1 \rangle)]^2 [1 - P_\ell(\langle x_0, x_2 \rangle)]} \right| \\
& \times \left| \left(\left\{ \partial_{\phi_{x_0}} P_\ell(\langle x_1, x_0 \rangle) - \partial_{\phi_{x_0}} P_\ell(\langle x_2, x_0 \rangle) \right\} C_{\ell;\phi\phi} [1 - P_\ell(\langle x_0, x_2 \rangle)] \right. \right. \\
& \left. \left. + \partial_{\phi_{x_0}} P_\ell(\langle x_2, x_0 \rangle) \left\{ C_{\ell;\phi\phi} [1 - P_\ell(\langle x_0, x_2 \rangle)] - [1 - P_\ell(\langle x_0, x_1 \rangle)] \partial_{\phi_{x_1}} \partial_{\phi_{x_2}} P_\ell(\langle x_1, x_2 \rangle) \right\} \right) \right| \\
& \leq \left| \frac{C_{\ell;\phi\phi}^2 P'_\ell(\langle x_1, x_0 \rangle)}{(1 - P_\ell(\langle x_0, x_1 \rangle))^2 (1 - P_\ell(\langle x_0, x_2 \rangle))} \right| \\
& \times \left(\left| (P'_\ell(\langle x_1, x_0 \rangle) (-\sin \theta_{x_1} \sin(\phi_{x_1} - \phi_{x_0})) - P'_\ell(\langle x_2, x_0 \rangle) (-\sin \theta_{x_2} \sin(\phi_{x_2} - \phi_{x_0}))) \right| \cdot \epsilon C_{\ell;\phi\phi} \right. \\
& \left. + \left| P'_\ell(\langle x_2, x_0 \rangle) (-\sin \theta_{x_2} \sin(\phi_{x_2} - \phi_{x_0})) \right| \cdot \left| (C_{\ell;\phi\phi} [1 - P_\ell(\langle x_0, x_2 \rangle)] \right. \right. \\
& \left. \left. - [1 - P_\ell(\langle x_0, x_1 \rangle)] \{ P''_\ell(\langle x_1, x_2 \rangle) \sin \theta_{x_1} \sin \theta_{x_2} \sin^2(\phi_{x_1} - \phi_{x_2}) + P'_\ell(\langle x_1, x_2 \rangle) \cos(\phi_{x_1} - \phi_{x_2}) \} \right) \right| \\
& \leq C_{\ell;\phi\phi}^2 M(\epsilon, \ell) \left(\left\{ |P'_\ell(\langle x_1, x_0 \rangle)| \cdot |(\sin \theta_{x_2} \sin(\phi_{x_2} - \phi_{x_0}) - \sin \theta_{x_1} \sin(\phi_{x_1} - \phi_{x_0}))| \right. \right. \\
& \left. + |(P'_\ell(\langle x_2, x_0 \rangle) - P'_\ell(\langle x_1, x_0 \rangle))| \cdot |\sin \theta_{x_2} \sin(\phi_{x_2} - \phi_{x_0})| \right\} \times \epsilon C_{\ell;\phi\phi} \\
& + M_1(\epsilon, \ell) C_{\ell;\phi\phi} |P_\ell(\langle x_0, x_1 \rangle) - P_\ell(\langle x_0, x_2 \rangle)| + M_1(\epsilon, \ell) |1 - P_\ell(\langle x_0, x_1 \rangle)| \\
& \times |C_{\ell;\phi\phi} - P''_\ell(\langle x_1, x_2 \rangle) \sin \theta_{x_1} \sin \theta_{x_2} \sin^2(\phi_{x_1} - \phi_{x_2}) - P'_\ell(\langle x_1, x_2 \rangle) \cos(\phi_{x_1} - \phi_{x_2})| \Big) \\
& \leq C_{\ell;\phi\phi}^2 M(\epsilon, \ell) \left(\epsilon C_{\ell;\phi\phi} M_2(\ell, \epsilon) (|\sin \theta_{x_2}| \cdot |\sin(\phi_{x_2} - \phi_{x_0}) - \sin(\phi_{x_1} - \phi_{x_0})| \right. \\
& + |\sin(\phi_{x_1} - \phi_{x_0})| \cdot |\sin \theta_{x_2} - \sin \theta_{x_1}| + M_3(\ell, \epsilon) |x_2 - x_1| \Big) \\
& + M'_1(\epsilon, \ell) |x_2 - x_1| + M''_1(\epsilon, \ell) \cdot |\sin \theta_{x_1} \sin \theta_{x_2}| \cdot \sin^2(\phi_{x_1} - \phi_{x_2}) \\
& + M''_1(\epsilon, \ell) \times |C_{\ell;\phi\phi} - P'_\ell(\langle x_1, x_2 \rangle) \cos(\phi_{x_1} - \phi_{x_2})| \Big) \\
& \leq C_{\ell;\phi\phi}^2 M(\epsilon, \ell) \left(\epsilon C_{\ell;\phi\phi} M_2(\epsilon, \ell) M_4(\epsilon, \ell) \cdot |\sin(\phi_{x_2} - \phi_{x_1}) - \sin(\phi_{x_1} - \phi_{x_0})| \right. \\
& + M_4(\epsilon, \ell) \cdot |\sin \theta_{x_2} - \sin \theta_{x_1}| + M_3(\epsilon, \ell) |x_2 - x_1| + M'_1(\epsilon, \ell) |x_2 - x_1| \\
& + M'''_1(\epsilon, \ell) \sin^2(\theta_{x_2} - \theta_{x_1}) + M''_1(\epsilon, \ell) \cdot |C_{\ell;\phi\phi} - P'_\ell(\langle x_1, x_2 \rangle)| \\
& \left. + M_1^{(iv)}(\epsilon, \ell) |1 - \cos(\phi_{x_2} - \phi_{x_1})| \right)
\end{aligned}$$

Now note that $C_{\ell;\phi\phi}$ is precisely equal to $P'_\ell(1)$, which can be rewritten as $P'_\ell(\langle x_1, x_1 \rangle)$. Replacing this in the last part of the above expression, we get the

following

$$\begin{aligned}
& \left| \frac{C_{\ell;\phi\phi}^2 \partial_{\phi_{x_0}} P_\ell(\langle x_1, x_0 \rangle)}{[1 - P_\ell(\langle x_0, x_1 \rangle)]^2 [1 - P_\ell(\langle x_0, x_2 \rangle)]} \right| \\
& \times \left| \left(\left\{ \partial_{\phi_{x_0}} P_\ell(\langle x_1, x_0 \rangle) - \partial_{\phi_{x_0}} P_\ell(\langle x_2, x_0 \rangle) \right\} C_{\ell;\phi\phi} [1 - P_\ell(\langle x_0, x_2 \rangle)] \right. \right. \\
& \left. \left. + \partial_{\phi_{x_0}} P_\ell(\langle x_2, x_0 \rangle) \left\{ C_{\ell;\phi\phi} [1 - P_\ell(\langle x_0, x_2 \rangle)] - [1 - P_\ell(\langle x_0, x_1 \rangle)] \partial_{\phi_{x_1}} \partial_{\phi_{x_2}} P_\ell(\langle x_1, x_2 \rangle) \right\} \right) \right| \\
& \leq C_{l\phi\phi}^2 M(\epsilon, \ell) \left(\epsilon C_{\ell;\phi\phi} M_2(\epsilon, \ell) M_4(\epsilon, \ell) \cdot |\sin(\phi_{x_2} - \phi_{x_1}) - \sin(\phi_{x_1} - \phi_{x_1})| \right. \\
& \quad + M_4(\epsilon, \ell) \cdot |\sin \theta_{x_2} - \sin \theta_{x_1}| + M_3(\epsilon, \ell) |x_2 - x_1| + M_1'(\epsilon, \ell) |x_2 - x_1| \\
& \quad + M_1'''(\epsilon, \ell) \sin^2(\theta_{x_2} - \theta_{x_1}) + M_1''(\epsilon, \ell) \cdot |P_\ell'(\langle x_1, x_1 \rangle) - P_\ell'(\langle x_1, x_2 \rangle)| \\
& \quad \left. + M_1^{(iv)}(\epsilon, \ell) |1 - \cos(\phi_{x_2} - \phi_{x_1})| \right) \\
& \leq K(\epsilon, \ell) |x_1 - x_2|
\end{aligned}$$

By replicating these set of calculations for each pair of terms in $\mathbb{E}(\widehat{f}_{j;q}^{x_0}(x_2) - \widehat{f}_{j;q}^{x_0}(x_1))^2$, we conclude that for every $x_1, x_2 \in B(x_0, \epsilon)$,

$$\mathbb{E}(\widehat{f}_{j;q}^{x_0}(x_2) - \widehat{f}_{j;q}^{x_0}(x_1))^2 \leq K(\epsilon, \ell) |x_2 - x_1|.$$

Next we wish to extend this to points inside the set $B(x_0, \epsilon) \setminus \{x_0\}$, but the Lipschitz coefficient K needs to be controlled. Observing that K depends on ϵ through the distance of points x_1, x_2 to x_0 , note that $\text{cov}(\widehat{f}_{j;q}^{x_0}(x_1), \widehat{f}_{j;q}^{x_0}(x_2))$ grows rapidly as either of x_1 or x_2 approach x_0 , whereas when x_1 and x_2 simultaneously approach x_0 , then the expression assumes the form of an indeterminate form, for which one can use the standard l'Hôpital's rule to get a precise form of the expression. Thus, let us first examine the following

$$\begin{aligned}
& \lim_{x \rightarrow x_0} \text{var}(\widehat{f}_{j;q}^{x_0}(x)) \\
& = \lim_{x \rightarrow x_0} \frac{1}{(1 - \rho(x_0, x))^2} \left(1 - \rho^2(x_0, x) + 2\rho(x_0, x) \partial_{\theta_{x_0}} \rho(x_0, x) \partial_{\theta_x} \rho(x_0, x) \partial_{\theta_x}^2 \rho(x, x) \right. \\
& \quad + 2\rho(x_0, x) \partial_{\phi_{x_0}} \rho(x_0, x) \partial_{\phi_x} \rho(x_0, x) \partial_{\phi_x}^2 \rho(x, x) + \{\partial_{\theta_{x_0}} \rho(x_0, x)\}^2 \{\partial_{\theta_x}^2 \rho(x, x)\}^3 \\
& \quad \left. + \{\partial_{\phi_{x_0}} \rho(x_0, x)\}^2 \{\partial_{\phi_x}^2 \rho(x, x)\}^3 \right)
\end{aligned}$$

Let us do the limit computations for just the first term of the variance

expression:

$$\begin{aligned}
& \lim_{x \rightarrow x_0} \frac{1 - \rho^2(x_0, x)}{(1 - \rho(x_0, x))^2} \\
&= \lim_{x \rightarrow x_0} \frac{-2\rho(x_0, x)\partial_{\theta_x}\rho(x_0, x)}{(-2)(1 - \rho(x_0, x))\partial_{\theta_x}\rho(x_0, x)} \\
&= \lim_{x \rightarrow x_0} \frac{\rho(x_0, x)\partial_{\theta_x}^2\rho(x_0, x) + (\partial_{\theta_x}\rho(x_0, x))^2}{(1 - \rho(x_0, x))\partial_{\theta_x}^2\rho(x_0, x) - (\partial_{\theta_x}\rho(x_0, x))^2} \\
&= \lim_{x \rightarrow x_0} \frac{\rho(x_0, x)\partial_{\theta_x}^3\rho(x_0, x) + 3\partial_{\theta_x}\rho(x_0, x)\partial_{\theta_x}^2\rho(x_0, x)}{(1 - \rho(x_0, x))\partial_{\theta_x}^3\rho(x_0, x) - 3\partial_{\theta_x}\rho(x_0, x)\partial_{\theta_x}^2\rho(x_0, x)} \\
&= \lim_{x \rightarrow x_0} \frac{\rho(x_0, x)\partial_{\theta_x}^4\rho(x_0, x) + \partial_{\theta_x}\rho(x_0, x)\partial_{\theta_x}^3\rho(x_0, x) + 3\partial_{\theta_x}^2\rho(x_0, x)\partial_{\theta_x}^2\rho(x_0, x) + 3\partial_{\theta_x}\rho(x_0, x)\partial_{\theta_x}^3\rho(x_0, x)}{(1 - \rho(x_0, x))\partial_{\theta_x}^4\rho(x_0, x) - 4\partial_{\theta_x}\rho(x_0, x)\partial_{\theta_x}^3\rho(x_0, x) - 3(\partial_{\theta_x}^2\rho(x_0, x))^2},
\end{aligned}$$

where we have applied l'Hôpital's rule at each step (four times), and we note that the final expression is indeed a nontrivial, determinate limit.

We note that we have assumed $\rho(x_0, x) = P_\ell(\langle x_0, x \rangle)$, and hence the derivatives above have the following form

$$\partial_{\theta_x} P_\ell(\langle x_0, x \rangle) = P'_\ell(\cdot) (\cos \theta_x \sin \theta_{x_0} \cos(\phi_x - \phi_{x_0}) - \sin \theta_x \cos \theta_{x_0})$$

$$\begin{aligned}
\partial_{\theta_x}^2 P_\ell(\langle x_0, x \rangle) &= P''_\ell(\cdot) (\cos \theta_x \sin \theta_{x_0} \cos(\phi_x - \phi_{x_0}) - \sin \theta_x \cos \theta_{x_0})^2 \\
&\quad + P'_\ell(\cdot) (-\sin \theta_x \sin \theta_{x_0} \cos(\phi_x - \phi_{x_0}) - \cos \theta_x \cos \theta_{x_0})
\end{aligned}$$

$$\begin{aligned}
& \partial_{\theta_x}^3 P_\ell(\langle x_0, x \rangle) \\
&= P'''_\ell(\cdot) (\cos \theta_x \sin \theta_{x_0} \cos(\phi_x - \phi_{x_0}) - \sin \theta_x \cos \theta_{x_0})^3 \\
&\quad + 2P''_\ell(\cdot) (\cos \theta_x \sin \theta_{x_0} \cos(\phi_x - \phi_{x_0}) - \sin \theta_x \cos \theta_{x_0}) (-\sin \theta_x \sin \theta_{x_0} \cos(\phi_x - \phi_{x_0}) - \cos \theta_x \cos \theta_{x_0}) \\
&\quad + P'_\ell(\cdot) (-\cos \theta_x \sin \theta_{x_0} \cos(\phi_x - \phi_{x_0}) + \sin \theta_x \cos \theta_{x_0})
\end{aligned}$$

$$\begin{aligned}
& \partial_{\theta_x}^4 P_\ell(\langle x_0, x \rangle) \\
&= P_\ell^{(iv)}(\cdot) (\cos \theta_x \sin \theta_{x_0} \cos(\phi_x - \phi_{x_0}) - \sin \theta_x \cos \theta_{x_0})^4 \\
&\quad + 3P'''_\ell(\cdot) (\cos \theta_x \sin \theta_{x_0} \cos(\phi_x - \phi_{x_0}) - \sin \theta_x \cos \theta_{x_0}) (-\sin \theta_x \sin \theta_{x_0} \cos(\phi_x - \phi_{x_0}) - \cos \theta_x \cos \theta_{x_0}) \\
&\quad + 2P''_\ell(\cdot) (\cos \theta_x \sin \theta_{x_0} \cos(\phi_x - \phi_{x_0}) - \sin \theta_x \cos \theta_{x_0})^2 (-\sin \theta_x \sin \theta_{x_0} \cos(\phi_x - \phi_{x_0}) - \cos \theta_x \cos \theta_{x_0}) \\
&\quad + 2P'_\ell(\cdot) (-\sin \theta_x \sin \theta_{x_0} \cos(\phi_x - \phi_{x_0}) - \cos \theta_x \cos \theta_{x_0})^2 \\
&\quad - 2P''_\ell(\cdot) (\cos \theta_x \sin \theta_{x_0} \cos(\phi_x - \phi_{x_0}) - \sin \theta_x \cos \theta_{x_0})^2 \\
&\quad + P''_\ell(\cdot) (\cos \theta_x \sin \theta_{x_0} \cos(\phi_x - \phi_{x_0}) - \sin \theta_x \cos \theta_{x_0}) (-\cos \theta_x \sin \theta_{x_0} \cos(\phi_x - \phi_{x_0}) + \sin \theta_x \cos \theta_{x_0}) \\
&\quad + P'_\ell(\cdot) (\sin \theta_x \sin \theta_{x_0} \cos(\phi_x - \phi_{x_0}) + \cos \theta_x \cos \theta_{x_0})
\end{aligned}$$

Thus we conclude that

$$\begin{aligned}
P_\ell(\langle x_0, x \rangle) \Big|_{x=x_0} &= 1 \\
\partial_{\theta_x} P_\ell(\langle x_0, x \rangle) \Big|_{x=x_0} &= 0 \\
\partial_{\theta_x}^2 P_\ell(\langle x_0, x \rangle) \Big|_{x=x_0} &= -P'(1) \\
\partial_{\theta_x}^3 P_\ell(\langle x_0, x \rangle) \Big|_{x=x_0} &= 0 \\
\partial_{\theta_x}^4 P_\ell(\langle x_0, x \rangle) \Big|_{x=x_0} &= 2P''_\ell(1) + P'_\ell(1)
\end{aligned}$$

Subsequently, we shall argue that by continuity, and the fact the field $\widehat{f}_{j;q}^{x_0}$ appears to be singular at x_0 , we conclude that for $x_1, x_2 \in B(x_0, \epsilon)$ and a small enough ϵ ,

$$\sup_{x_1, x_2 \in B(x_0, \epsilon)} \mathbb{E} \left(\widehat{f}_{j;q}^{x_0}(x_2) - \widehat{f}_{j;q}^{x_0}(x_1) \right)^2 = \lim_{(x_1, x_2) \rightarrow (x_0, x_0)} \mathbb{E} \left(\widehat{f}_{j;q}^{x_0}(x_2) - \widehat{f}_{j;q}^{x_0}(x_1) \right)^2$$

The limit on the RHS can again be evaluated by applying l'Hôpital's rule, and thus, the (uniform) Lipschitz behavior is justified. Thereafter, we note that by the isotropy of the underlying field $f_{j;q}$, the $\mathbb{E} \left(\sup_{x \in S^2 \setminus \{x_0\}} \widehat{f}_{j;q}^{x_0}(x) \right)$ does not depend on x_0 , and thus we get a uniform (over j and x_0) Lipschitz bound, as claimed.